THE OPEN UNIVERSITY
A SCIENCE FOUNDATION COURSE

UNIT 2 MEASURING THE SOLAR SYSTEM

STUDY GUIDE

This Unit has three major components: the text, an experiment, and the TV programme 'Measuring—the Earth and the Moon'.

Although this Unit is ostensibly about the Earth and its position in the Solar System, it is also designed to introduce you to a number of mathematical skills that you'll need to use many times during the Course. Consequently, the text of the Unit is laid out very much in the manner of a workbook: you are asked to fill in blanks, complete Tables, draw graphs and, of course, answer lots of questions. In fact, the motto of this Unit might well be 'learning by doing'.

In order to give you plenty of practice in mastering a new skill, its first introduction in the text is usually followed immediately by one or more questions. All these questions are labelled as ITQs, although, since they also enable you to test yourself as you go along, they actually serve as ITQs and SAQs rolled into one. For this reason, there are no separate SAQs in this Unit.

The main thrust of the story also includes an important experiment, in which you will measure the distance to the Moon. This experiment, and the activities leading up to it, are to be found in Section 3.4; if you want a quick preview of the apparatus you will need to assemble (some of which you must provide yourself), look at Section 3.4.1. Although you won't be able to *analyse* the results of your experiment until you have read to the end of Section 3.4, you can do the *practical* part of the work whenever this is convenient. You should therefore **seize the opportunity of the first cloud-free night to make the measurements**; it isn't essential that the Moon be full, but the experiment is easiest to perform within a day or two either side of a full Moon.

The TV programme, 'Measuring—the Earth and the Moon' is associated mainly with Section 3. Notes on the TV programme are given at the end of the text. In order to derive most benefit from this programme, you must have read to at least the end of Section 3.4 before watching it. Part of the programme is used to explain how to make an important correction to one of the measurements you take in the experiment. However, you should not postpone doing the experiment until after you have seen the programme—do the experiment at the first available opportunity. The TV programme simply describes how to apply a correction to your data to make the result more accurate—it neither invalidates your measurements nor provides any extra information on the experimental procedure.

1 INTRODUCTION

This Unit is about measurement—about the assumptions made, the reasoning used and the limitations involved, whenever you make a measurement. In order to understand this topic, you must do more than simply listen to what other people have to tell you. You must learn to make measurements for yourself, and you must work to understand why other scientists have chosen to make the measurements they have made. That is why, during the course of this Unit, you will be asked to carry out your own experiment, compile Tables of results, and criticize constructively other people's measurements. Rather than ask you to learn these techniques and skills as an end in themselves, the Course Team has tried to weave the techniques into the story of the early discovery of the 'rules' governing the Solar System. Because you will be *using* scientific skills in the context of this fascinating story, and have the opportunity to practise them as you go along, you should find yourself encouraged and motivated to acquire the necessary expertise.

Many of these skills require mathematics. Making and interpreting measurements often involves using mathematical reasoning. For instance, a *direct* measurement of the distance between the Earth and the Sun is not possible. So an *indirect* approach must be adopted. This may well mean measuring angles (so you must know about angular measure) and then deducing the distance using a 'triangulation' method (so you must know something about the properties of triangles). These mathematical tools are introduced as and when they are needed. You may find that the explanation of them in this Unit is quite adequate for you. If so, you probably won't find the Unit too difficult. But if you're not very familiar with the mathematics, please make sure that you take the advice, given at various places throughout the text, to refer to the explanations and examples in the relevant Sections of *Into Science*.*

You may be wondering why it is necessary to bother with this *quantitative* aspect of science at all. Why not just concentrate on the ideas and concepts of science? Well, in some areas we do just that. Most of Unit 1, for example, was *qualitative* in approach, and there was plenty of 'good science' in that. But most science can't stop at the qualitative level. Sooner or later, two or three qualitative explanations of a phenomenon will come into conflict. And if science is to be a little more objective than the 'my-explanation-is-as-good-as-yours' argument will allow, there has to be some way of choosing between these rival explanations. That is where measurement becomes important.

Of course, the desire to choose between one scientific model and another does not have to be the sole motivation for making measurements! Indeed, people have long realized that there are more down-to-earth reasons why they should be able to measure things. How else, for instance, could they know the quantity of grain they had for sale, or the distance they would have to carry it to market, or the time it would take to walk there? No, the need to measure things existed long before the need to differentiate between contending theories about the Solar System.

One consequence is that we have inherited a variety of measurement standards. Naturally, these standards have evolved over the centuries, from the rather crude and variable measures of primitive people, to the highly precise measurement standards used in modern science. We no longer measure distances in terms of a man's stride: we now have length standards that are accurate to 1 inch in 10 000 miles. And the immensely impressive 'stone calendar' at Stonehenge (Figure 1) has been superseded by an atomic clock which operates with an accuracy of 1 second in 30 000 years. Indeed, this evolution of measurement standards has been absolutely essential for the development of science. For without a system of well-defined units and precise, reproducible standards, scientists would find it impossible to com-

* The Open University (1993) *Into Science*, The Open University.

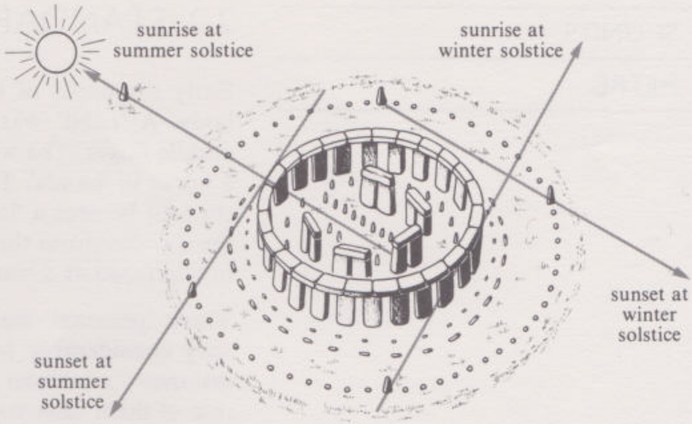


FIGURE 1 Stonehenge. The alignment of various stones is related to the position of the rising and setting Sun on the days of the summer and winter solstice. Sunrise on the summer solstice appears to have been of particular importance; the alignment is such that this sunrise has to be viewed from the altar stone, inside the monument.

municate their findings to one another unambiguously. It is therefore essential that you become acquainted with the international system of units and standards in present-day use. That is why the whole of Section 2 is devoted to a discussion of these standards. However, you should not try to memorize all the details given in this Section. Instead, you should view it as providing background information to help you put your understanding of the measurement techniques described in the rest of the Course into the context of modern science and technology.

The story of the measurement of the Solar System (which takes up most of the remainder of the Unit) is divided into three parts. The first part (Section 3) describes how the early Greek astronomers estimated the sizes and distances involved in our Sun–Earth–Moon system. Section 4 is basically concerned with measurements involving the other planets of the Solar System, and in particular with the pioneering work of Copernicus, Tycho Brahe, and Kepler. The culmination of this work was the formulation, by Kepler, of his 'laws of planetary motion'—three laws that succinctly summarize the *regularity* of the motion of the planets in the Solar System. An epilogue to this story of Kepler's discovery of his three laws is provided in Section 5. Galileo, using the newly invented telescope, discovered four moons circulating around Jupiter. Kepler's laws were applicable to the planets orbiting the Sun. Could they also be applied to the moons orbiting Jupiter? If so, what was the 'mechanism' behind these laws? You will not find the answer to this last question in Unit 2. The 'explanation' for Kepler's laws was provided by Newton, and Newton's contribution to scientific thought is the main focus of Unit 3.

2 MEASUREMENT: UNITS AND STANDARDS

2.1 SETTING THE STANDARD

Measurement can usually be reduced to a comparison of something of interest with some agreed standard. For instance, if you were to take the width of this book as a unit of length, the lengths of all other objects could be related to this width. One object may be twice as long as the width of the book (i.e. two 'book-widths'); a second may be three-and-a-half times as long (3.5 book-widths); another may be a quarter as long (0.25 book-widths). Once the unit is established, measurement becomes simply a matter of counting. So, if you were told that the Walton Hall boiler-house is 60 book-widths high, you would have some idea of how tall it is, even though you may never have seen it! But note that it was important for you to know what the unit of measurement was; furthermore, you even had access to a 'copy' of it. Someone who did not know what the unit of measurement was, or who did not have access to the 'standard' book-width, would have no idea what was meant by a height of 60 units. We need standards, and these standards must be known by everyone with whom we wish to communicate. But what standards should we choose?

SI UNITS

METRE

2.2 STANDARDS OF LENGTH

Early standards of length tended to be related to the size of the human body. A 'cubit' was the distance between the elbow and the tip of the middle finger. The width of a man's hand was used to express the height of a horse in 'hands'. The width of the thumb became an 'inch', the length of the foot became a 'foot'. When measuring cloth, it was convenient to define the distance from the nose to the end of the middle finger when the arm was outstretched as a 'yard'.

These 'personal' standards, though conveniently portable, can, however, vary considerably from person to person. Get two drapers together, for instance—a tall one and a short one—and, with this definition of the yard, one of them sells you 'cheaper' cloth! Clearly, we need a greater degree of standardization than this. So what do we do? We invent the *yardstick*. Copies of this stick can then be sent all over the country, with the result that a yard of cloth in Edinburgh is more or less the same length as a yard of cloth in London. But what do we mean by 'more or less?' To within $\frac{1}{10}$ inch? Nobody is likely to grumble about that sort of uncertainty if they are buying cloth; but in science we may well want to measure to smaller subdivisions than that. So the 'precision' of our standard must improve. The ultimate limit to the accuracy with which we can make a measurement (no matter how good the equipment) will always be determined by the uncertainty in the standard. It is no good giving a measurement to an accuracy of $\frac{1}{100}$ inch, if the agreement as to what constitutes a standard yard is only good to $\frac{1}{10}$ inch. (Mind you, once you have found a way of measuring to an accuracy greater than that to which the standard is defined, the onus is on you to petition for a new standard based on your improved technique!) Hence, as scientists have sought to make more and more accurate measurements, so they have also had to devise more and more precise measurement standards.

Nowadays, there is also a further requirement: our measurement standards should be international. Clearly, this was unimportant when different communities did not interact with one another. Today, however, if scientists in Europe are to understand the measurements of scientists in the USA, the USSR, or Japan, they must be acquainted with one another's standards of measurement. Better still, they should all use the same standards. In 1960 it was formally agreed to standardize according to the 'Système International d'Unités'. This system, usually abbreviated as **SI units**, is now in universal use in the scientific community and is the one with which you will be primarily concerned in this Course. The SI system is *metric*, i.e. based upon the **metre** as the standard unit of length.

The history of the metric system dates back to Napoleonic France, where the metre was originally defined as one ten-millionth of the distance from the Equator to the North Pole along a meridian passing through Dunkirk and Barcelona, and hence also passing very close to Paris (Figure 2). This choice of standard was a very 'safe' one—the standard could not be lost! But it could hardly be called practical. So in 1889 it was officially decided to define the metre as the distance between two parallel marks inscribed on a particular bar made of the metal alloy platinum–iridium. (This alloy was chosen because of its exceptional hardness and its resistance to corrosion.) Furthermore, to ensure the reproducibility of measurements of this length, the bar had to be kept under specific conditions. For instance, it had to be supported in a particular way, so as to minimize deformations, and it had to be kept at the temperature of melting ice so as to prevent the expansion or contraction that would occur if the temperature were allowed to vary. This *standard metre* still exists. It is housed in the International Bureau of Weights and Measures in Sèvres, near Paris. Copies of this bar, i.e. *secondary standards*, have been made and distributed to national standards offices throughout the world.

Even this, however, was not completely satisfactory. For although the length of an object could be compared with the standard metre to a precision of about two parts in ten million (by using a high-powered microscope

FIGURE 2 The metre was originally defined as one ten-millionth of the distance from the North Pole to the Equator. A team of French surveyors was engaged to measure the distance between Dunkirk and Barcelona. The length of the full quadrant was then determined from astronomical measurements of latitude.

to view the finely inscribed marks on the metre bar), this precision was still inadequate for some scientific purposes. (Don't forget, the precision to which we know the standard must be better than the *most* accurate measurement we wish to make.) In addition, making comparisons with a bar that had to be kept under specific conditions in a standards laboratory was still inconvenient. What was required was a standard that anyone (or at least, any scientist) could have access to in any laboratory, a standard that did not require copies to be made (and hence eliminated the problem of inexact copies), and a standard that could be relied upon never to change.

In 1961, by international agreement, a new standard of length was defined, based on the wavelength of a particular colour of light. Think, for example, of a sodium street lamp. No matter where the street lamp is situated—London or Glasgow, New York or Helsinki—it still emits the same yellow-coloured light. As you will see in Units 11–12, this is because the colour of the light is determined by the 'structure' of the sodium atoms—and sodium atoms are the same the world over. The scientific way of describing the colour of a light is in terms of its 'wavelength', which you will meet in Unit 10. In fact the substance chosen in connection with the length standard was not sodium, but krypton, whose atoms emit light of a characteristic orange-red colour.

So the length of the standard metre bar was carefully measured in terms of this 'wavelength of krypton light' and it was agreed that *exactly* 1 650 763.73 wavelengths would constitute one metre. This number was chosen so that the old definition of the metre (as the distance between the two inscribed marks on the platinum–iridium bar), and the new definition of the metre (as a particular number of wavelengths of krypton light) were kept in agreement. The advantage of the new definition was that it provided a standard of length far more precise than the metre bar. In addition, the krypton standard was readily available to laboratories all over the world, since krypton is present, albeit in small amounts, in the Earth's atmosphere.

The krypton wavelength is the sort of unit that can be called a *natural* unit since it depends on a particular natural property—namely, the fact that all atoms of a particular species are identical, and consequently always emit light of exactly the same colour.

Even then, that wasn't the end of the story. During the 1970s, it became clear that measurements of length made with the latest technology were limited not by the measuring techniques themselves, but rather by the definition of the metre in terms of the krypton wavelength. In other words, the new measuring techniques were potentially *more* precise than the standard of length. So, in 1983, it was agreed to adopt yet another definition of the

metre. This one was different from all those that had gone before, because instead of being based on a *measurement*, the new standard was tied to a *defined* value. The property chosen this time was the speed of light in a vacuum.

Einstein, in his special theory of relativity, said that the speed of light in a vacuum (i.e. in empty space) is the maximum speed at which energy (or matter) can be transferred from one place to another: nothing can travel faster than light. Physicists are now convinced that the speed of light in a vacuum is a fundamental constant—always the same, everywhere in the Universe—and therefore perfect for use as a basic standard of measurement.

Surely this is contradiction

So nowadays, the speed of light in a vacuum is a *defined* quantity: in one second light travels *exactly* 299 792 458 metres. If you're wondering why scientists didn't take advantage of what seems a golden opportunity to define the speed of light as a nice round number, the explanation is quite simple. They wanted to keep the new definition completely consistent with the previous, krypton-wavelength standard. The slight penalty for that is having to live with an unwieldy value for the speed of light.

2.3 STANDARDS OF TIME

The concept of length is relatively straightforward. It is essentially a geometrical concept—a distance between two points in space. Length is easy to measure, too. We can make a metre rule; we can move the metre rule from place to place; we can use the metre rule today and tomorrow and the next day; and we can measure a particular length as often as we like.

Time is a more difficult quantity to measure. An interval of time can be used only once, and then it's gone—unless, that is, we can find some process that repeats with a regular and countable pattern. You met such a process in Unit 1: the cycle of day following night. Unfortunately, there are slight variations in the Earth's orbital speed during the course of a year. These variations, in turn, cause the interval between successive culminations of the Sun also to vary throughout the year. That's hardly satisfactory from the point of view of standardization—it means that a 'summer day' and a 'spring day' do not correspond to quite the same time interval. So we must look for a better standard. One possibility is to choose the *mean solar day* as the standard. The mean solar day is the *average* (taken over a year) of the time the Earth takes to spin once on its axis, relative to the Sun. The division of this mean solar day into 24 hours, and each hour into 60 minutes, and each minutes into 60 seconds, then gives us the basic unit of time: the **second**. It is the second that has been adopted as the SI unit of time.

This way of defining a unit of time in terms of the solar day has been adequate for the majority of everyday applications, but it has proved unsatisfactory for very high precision work. For, in addition to the variation in the solar day caused by variations in the Earth's orbital speed, there is also a cumulative slowing down of the Earth's spin (probably caused by tidal friction), the net effect of which is to cause our solar clock to lose 15 milliseconds every 1 000 years.

So in 1967, a *natural unit of time* was adopted. Like the natural unit of length, this natural unit of time was based upon the identical nature of all the atoms of a particular species—in this case, the atoms of caesium. Every atom vibrates at a characteristic frequency. The second is now defined as the time required for a caesium atom to vibrate exactly 9 192 631 770 times. The world's first caesium clock (Figure 3) was developed at the National Physical Laboratory, Teddington, in 1967. It kept time to an accuracy of better than 1 second in 10 000 years. Current technology, however, is doing even better than this. There are now clocks capable of providing a precision of 1 second in 3 million years!

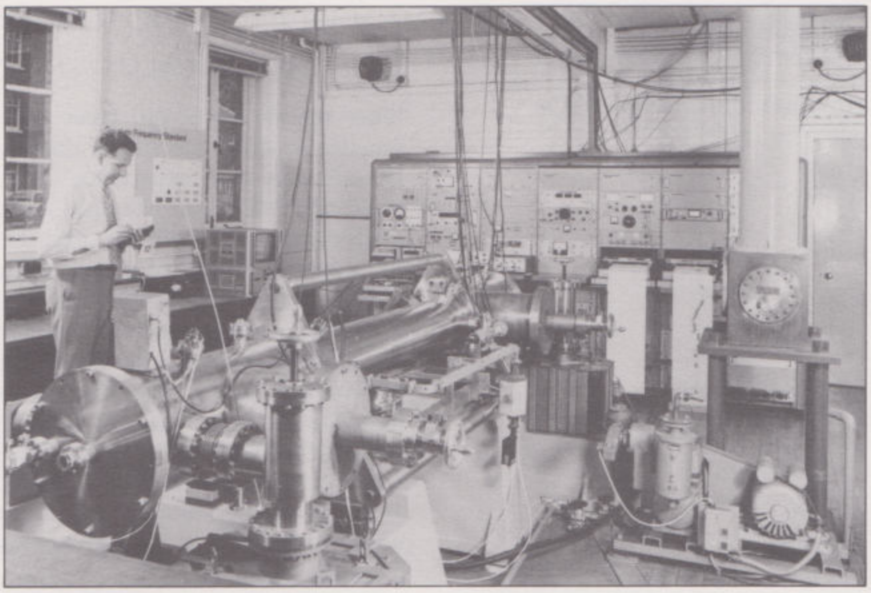


FIGURE 3 The world's first caesium clock. It was not exactly designed for domestic use, and did not have a conventional 'clock face', but it did have the virtue of keeping time to better than 1 second in 10 000 years!

2.4 STANDARDS OF MASS

The concept of mass is quite a difficult one. Intuitively, we tend to regard mass as a measure of the amount of matter in an object. However, as we shall see in Unit 3, if one tries to work out a theory of the way in which objects move and interact using that kind of definition of mass, problems soon arise. In fact, a proper definition of mass requires an understanding of the physics of moving objects.

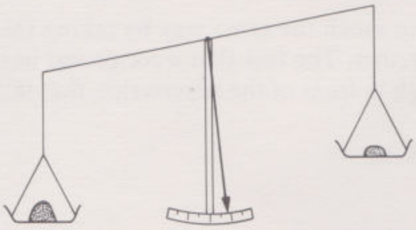
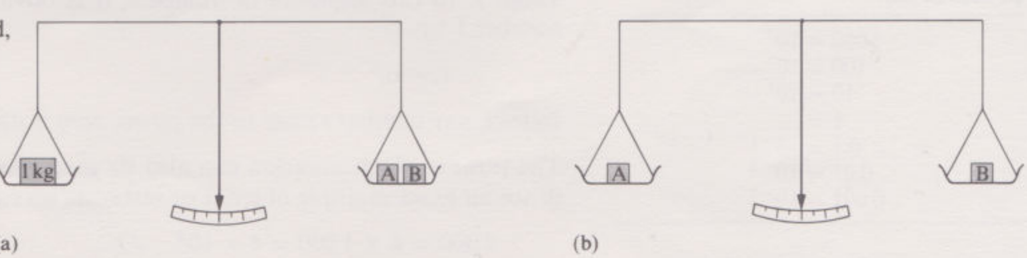


FIGURE 4 Schematic diagram of a beam balance used for comparing masses.

Fortunately for our purposes here, none of this creates a barrier to defining a standard of mass. What has been agreed is that the mass of one particular lump of matter will be called one **kilogram** and that this will define the SI unit of mass. (The internationally agreed standard lump is actually a cylinder of platinum–iridium kept in the International Bureau of Weights and Measures at Sèvres.) We can then compare this standard lump with any other lump of matter by using, for example, a beam balance (Figure 4). When the beam is level, we *define* the mass of the second lump to be the same as that of the standard. Secondary standards of mass made in this way have been distributed to standards laboratories throughout the world.

Of course, once a primary standard kilogram has been decided upon, it is possible to make a whole range of secondary standards—not just for one kilogram, but also for 0.5 kg, 2 kg, 5 kg, 10 kg, etc. The procedure is outlined in Figure 5.

FIGURE 5
Generating a 0.5 kg standard, given a 1 kg standard. When the beam balances with *both* the arrangements of masses shown in (a) and (b), then:
mass of A
= mass of B
= 0.5 kg



Naturally, it would be good if we could find an atomic standard of mass to supersede this operational standard. Such a standard does exist for the comparison of the masses of individual atoms, as you will find out in Units 11–12, but unfortunately we have not, as yet, discovered a way of scaling up this atomic mass standard with sufficient precision to allow us to use it for everyday mass comparisons. But doubtless we shall, one day!

POWERS-OF-TEN (SCIENTIFIC) NOTATION

ORDER OF MAGNITUDE

2.5 UNITS AND DIMENSIONS

2.5.1 THE POWERS-OF-TEN NOTATION

Since the choice of our basic units is essentially arbitrary, it would seem sensible to make the size of these units reflect the scale of things, or events, in our everyday experience. The metre, second and kilogram do this reasonably well. For example, a metre is about the size of a man's stride, a second approximately the time between consecutive human heartbeats, a kilogram the mass of a bag of sugar. However, when you consider that science is concerned with the whole range of natural phenomena—from events on a subatomic scale to developments on a galactic scale—you can see that we shall often meet quantities that are either very much smaller, or enormously larger than the size of these basic units. For instance, the distance to our nearest-neighbour star, Alpha Centauri, is about 40 400 000 000 000 000 metres. Or, at the other extreme, the time a radio signal takes to travel from London to Edinburgh is about 0.001 68 seconds. It is clearly inconvenient to have to write these quantities in this way, so scientists use a neat kind of shorthand—**powers-of-ten notation** (often called **scientific notation**).

The basic idea behind this notation is that several tens multiplied together generate very large numbers. In the powers-of-ten notation these numbers are represented by placing a superscript after the number 10; the superscript indicates the number of tens that have to be multiplied together to get the number. That is:

10^1 means $10 = 10$

10^2 means $10 \times 10 = 100$

10^3 means $10 \times 10 \times 10 = 1\,000$

10^4 means $10 \times 10 \times 10 \times 10 = 10\,000$

10^5 means $10 \times 10 \times 10 \times 10 \times 10 = 100\,000$

This superscript is called the *power* to which ten is raised.

We can cope with very small numbers in much the same way by taking the *reciprocal* of several tens multiplied together. The fact that a reciprocal has been taken is indicated by a negative sign in front of the superscript, that is:

$$0.1 = \frac{1}{10} = \frac{1}{10^1} \text{ is written as } 10^{-1}$$

$$0.01 = \frac{1}{100} = \frac{1}{10^2} \text{ is written as } 10^{-2}$$

$$0.001 = \frac{1}{1\,000} = \frac{1}{10^3} \text{ is written as } 10^{-3}$$

and so on.

With this notation, the number 1 can also be accommodated. Look at Table 1. In this sequence of numbers, it is obviously logical to define the number 1 by

$$1 = 10^0$$

Indeed, *any* number raised to the power zero is defined to be equal to 1.

TABLE 1 Numbers expressed as powers of ten

$1\,000 = 10^3$
$100 = 10^2$
$10 = 10^1$
$1 = \ldots$
$0.1 = 10^{-1}$
$0.01 = 10^{-2}$
$0.001 = 10^{-3}$

The powers-of-ten notation can also be used when the number in question is not an exact multiple of ten. For instance, we can write:

$$4\,000 = 4 \times 1\,000 = 4 \times 10^3$$

Similarly:

$$0.06 = 6 \times (1/100) = 6 \times 10^{-2}$$

Consequently, we can now write the distance to Alpha Centauri as 4.04×10^{16} m, and the time for the radio signal to travel from London to Edinburgh as 1.68×10^{-3} s. These numbers are much more easily assimilated in this form than in the long form in which they incorporate a large number of zeros.

ITQ 1 Use scientific notation to express the number of seconds in one solar day (i.e. 24 hours).

8.64×10^4

Because scientists frequently use this kind of notation, they have devised a system of names and abbreviations for some of the powers of ten to be applied as prefixes to the basic units of measurement (Table 2). Thus one thousand metres can be said as one kilometre and written as 1 km. One thousandth of a second is said as one millisecond and written as 1 ms.

TABLE 2 Prefixes to units of measurement

Prefix*	Symbol	Equivalent of prefix in powers-of-ten notation
tera	T	10^{12}
giga	G	10^{9}
mega	M	10^{6}
kilo	k	10^{3}
centi	c	10^{-2}
milli	m	10^{-3}
micro	μ	10^{-6}
nano	n	10^{-9}
pico	p	10^{-12}
femto	f	10^{-15}

* Note that when a prefix is placed in front of a unit, in effect it produces a new unit. Consequently, km^2 (for instance) should be read as $(\text{kilometres})^2$ and not as kilo × $(\text{metres})^2$; thus $1\,km^2 = (10^3\,m)^2 = 10^6$ square metres.

As you can see from Table 2, the prefixes generally change in steps of 10^3 (i.e. 1 000). This is the preferred SI convention. However, the intermediate prefix *centi* is such common usage that it has also been included in the Table. You will need the information in Table 2 many times during the Course, so for ease of reference it is reprinted on the back cover of the Glossary.

□ Light takes approximately 3.34×10^{-7} seconds to travel 100 metres. What is this time in nanoseconds?

■ Because 1 nanosecond $= 10^{-9}$ seconds,

it follows that

$$1 \text{ second} = 10^9 \text{ nanoseconds}$$

Hence

$$3.34 \times 10^{-7} \text{ seconds} = (3.34 \times 10^{-7}) \times 10^9 \text{ nanoseconds}$$
$$= 334\,\text{ns}$$

2.5.2 ORDERS OF MAGNITUDE

Scientists sometimes call a power of 10 an **order of magnitude**. So, for instance, we could say that £1 is an order of magnitude more valuable than a 10p piece, or *two* orders of magnitude more valuable than a 1p piece. However, it is more common for this 'order of magnitude' expression to be used in an *approximate* sense. For example, since the distance from London to Aberdeen is about 490 miles and the distance from London to Milton Keynes about 55 miles, we might say that Aberdeen is an order of magnitude further away from London than is Milton Keynes. For many purposes this kind of statement would be quite accurate enough.

Similarly, in science, it is often very useful to be able to get just a rough idea of the size of some quantity, without having to do an exact calculation or to carry out a very careful experiment. In fact, the idea of quoting quantities to 'within an order of magnitude' is so useful that a special symbol has been devised to represent this kind of relationship. We write: the distance to Alpha Centauri $\sim 10^{16}$ m. The symbol $\sim$ means 'is of the same order of magnitude as'. This statement tells us that the distance involved is 10^{16} m *to within a factor of ten.*

DIMENSIONS

□ How could we calculate, to within an order of magnitude, the number of seconds in the lifespan of a typical adult in Britain?

■ We can do this quite quickly by making approximations to get numbers that are easy to handle. If we were to assume a typical lifespan of 70 years, the full calculation would be

70 years × $365\frac{1}{4}$ days per year × 24 hours per day × 60 minutes per hour × 60 seconds per minute

If you were to work this out on your calculator you would find it came to 2 209 032 000 seconds. Expressed as an order of magnitude (i.e. to the nearest integral power of ten) this is 10^9 s.

However, there is no point in making such an accurate calculation if all we want is an order of magnitude estimate. It is much simpler to approximate, indeed very crudely, so as to make the numbers as easy as possible:

$$70 \text{ years} \approx 50 \times 400 \times 25 \times 60 \times 60 \text{ seconds} = 1\,800\,000\,000 \text{ s}$$

As an order of magnitude, this too is 10^9 s.

The ability to make this sort of estimate is a very useful one for a scientist. In fact, it is a good habit to check the result of every calculation by doing a rough approximation, just to ensure that the answer you get is sensible.

TABLE 3 The meaning of some mathematical symbols

Symbol	Meaning
$=$	is equal to
$\approx$	is approximately equal to
$\sim$	is of the order of magnitude of
$>$	is greater than
$<$	is less than
$\gtrsim$	is greater than or roughly equal to
$\lesssim$	is less than or roughly equal to
$\geqslant$	is greater than or equal to
$\leqslant$	is less than or equal to

ITQ 2 You calculated in ITQ 1 that there are 8.64×10^4 seconds in one day. How may seconds are there, *to within an order of magnitude*, in 1 week?

$\sim 6.0 \times 10^5$

The mathematical symbols that you are likely to come across in this Course are listed in Table 3. You will have already met the familiar 'is equal to' sign. The $\approx$ sign, meaning 'is approximately equal to' is not quite as loose as the $\sim$ sign. It implies the *rounding off* of a quantity, rather than a possible factor-of-ten uncertainty. For example, one could write

$26.7 \approx 27$ or $267 \approx 270$

but $267 \sim 300$

2.5.3 DIMENSIONS

Before leaving this subject of units and standards, there is one final point that is worth mentioning: *every measured quantity must have units associated with it.* Making a measurement implies comparing the quantity being measured with some agreed standard, so the measured quantity must take on the same units as the standard. This is a point you should take to heart: wherever you write down the numerical value of a measurement, it is essential always to write down the units as well.

Note the importance of the word 'measured' in the previous paragraph. Directly measurable quantities have units, but their ratios and multiples do not necessarily do so. For instance the ratio of two lengths is a pure number:

$$\text{e.g. } \frac{50\text{ m}}{25\text{ m}} = 2 \quad \text{and} \quad \frac{50\text{ km}}{25\text{ m}} = \frac{50 \times 1\,000\text{ m}}{25\text{ m}} = 2\,000$$

These apparently trivial examples are important because they show that units can be multiplied and divided by one another. We make use of this fact whenever we read the speedometer in a car. If you've driven steadily along the motorway and noted that the time taken between successive mile markers was 1 minute, you'd instantly calculate your speed as:

$$\frac{1\text{ mile}}{1\text{ minute}} = \frac{60\text{ miles}}{60\text{ minutes}} = \frac{60\text{ miles}}{1\text{ hour}} = 60\text{ m.p.h.}$$

Thus whenever you divide one quantity by another, you divide not only the numbers but also their respective units. So, 10 kilometres in 5 hours is 2 km/hour (i.e. 2 kilometres per hour). Similarly, whenever you multiply two quantities together, you must also multiply their respective units. So, 5 metres × 12 metres is 60 metres^2.

□ What is $3\,\text{mm} \times 2\,\text{m}$, in units of m^2?

■ $$3\,\text{mm} = 0.003\,\text{m}$$

So

$$(3\,\text{mm}) \times (2\,\text{m}) = (0.003\,\text{m}) \times (2\,\text{m}) = 0.006\,\text{m}^2$$

Alternatively, we can reason that

$$3\,\text{mm} = 3 \times 10^{-3}\,\text{m}$$

So

$$(3\,\text{mm}) \times (2\,\text{m}) = (3 \times 10^{-3}\,\text{m}) \times (2\,\text{m}) = 6 \times 10^{-3}\,\text{m}^2$$

The importance of all this is that it provides us with a very powerful tool for checking whether an equation relating two physical quantities is likely to be correct—the technique called *dimensional analysis*. This technique is based on a recognition of the fact that we can only equate two quantities if they are both lengths, or both speeds, or both times or whatever—we can only equate like with like. The scientist's way of saying this is that the quantities must have the same **dimensions**.

Why the special word 'dimensions' instead of just 'units'? The reason is quite simple: the use of dimensions allows us to equate quantities expressed in units that differ only by a conversion factor. For example, feet, miles and metres are all different units, yet they have a common dimension—length. Similarly, hours and seconds, though different units, both have the dimension of time. To a good approximation:

$$50\,\text{m.p.h.} = 22.35\,\text{m/s}$$

This equation, which balances miles per hour against metres per second, is perfectly valid (provided, of course, that the appropriate conversion factor has been incorporated into the equation). For although the units do not match exactly, the dimensions do: both sides of the equation have dimensions of length divided by time.

SUMMARY OF SECTION 2

In this Section, you have been introduced to some of the scientific and mathematical skills and techniques that you will need later in this Unit, as well as in subsequent Units. In particular, you have seen how to:

1 handle reciprocals, multiples and fractions of quantities;

2 combine quantities by multiplying or dividing one by another;

3 express a number using the powers-of-ten notation;

4 convert the units of a physical quantity to other equivalent units using (where possible) standard prefixes and powers of ten;

5 use the order-of-magnitude symbol appropriately;

6 identify the basic dimensions of a quantity, and so check the validity of equations involving that quantity.

You have also seen that any measured quantity must have units associated with it, and have been introduced to the measurement standards and SI units for length, time and mass (the metre, the second and the kilogram, respectively).

3 THE EARTH, THE SUN AND THE MOON

3.1 INTRODUCTION

Let us now turn from the theory of measurement to its practice. In Unit 1 you saw how our present-day model of the Solar System was in qualitative agreement with observation. But, of course, the model of the Solar System was not developed purely on the basis of qualitative observations. Historically, qualitative observations were often mixed in with measurements—measurements that were sometimes accurate and sometimes wildly inaccurate (when viewed with hindsight). Sometimes the inaccurate measurements actually held back the development of the model. Why were some of the measurements so inaccurate? As you might guess, the answer to this question is not simple. In part, the inaccuracies can be attributed to poor tools and observational aids. A lot of the difficulty, however, also lay in the indirect way in which the measurements were made, or in the rather cavalier manner in which the data were extended beyond the range of the actual measurements. Sometimes dubious assumptions were used in calculating the required quantity from the measurements.

This, however, is not to devalue the work of our ancestors! Indeed, the tricks they used then are, in many ways, very similar to the tricks we use ourselves today. Whenever the measurement to be made lies outside the scope of current observational techniques, we have to rely on our initiative to devise and interpret the results of indirect methods of measurement. Initiative was something the early 'measurers of the Solar System' had in abundance. So perhaps we can learn something by looking at the sort of measurements they made, and the sort of reasoning they employed.

3.2 THE SIZE OF THE EARTH

3.2.1 THE SCIENTIFIC SCHOOL AT ALEXANDRIA

The city of Alexandria was founded at the mouth of the River Nile by Alexander the Great, during his conquest of Asia Minor, Egypt and Persia. It became an important centre of learning during the fourth, third and second centuries BC, and the Museum of Alexandria (which functioned like an academy and university) attracted some of the foremost Greek scholars of the time. In about 300 BC, a school of astronomy was established there, a school that was to bring a new attitude to the science. For these astronomers sought to *quantify* the sizes and distances involved in the Solar System and so to turn astronomy into a 'real' science. Conjecture was all very well, they argued, but it must be based on measurement.

The obvious starting point was to try to find the size of the Earth. How big was the world they lived on? One of the earliest estimates was provided in about 235 BC by the Greek astronomer Eratosthenes. His measuring technique was, of necessity, indirect. After all, very little of the world was known to the Greeks in 235 BC. And, of course, Eratosthenes did not have the advantage of our modern technological aids such as radio communications and space flight. Yet, starting from the assumption that the Earth was spherical,* he managed to get a value for the circumference (and thus also for the radius) of the Earth that compares very favourably with the currently accepted value. (He was probably within 5% of the modern value.)

* The idea of a spherical Earth was well established in Greek culture. Aristotle had argued that, on the grounds of symmetry alone, the Earth *must* be a sphere. But there was also the experimental evidence provided by (a) the always circular shape of the Earth's shadow thrown onto the Moon during a lunar eclipse, and (b) the change in the position of the stars as an observer travelled northwards or southwards (see Unit 1). It was not until the Middle Ages that the 'flat Earth' idea again became popular.

3.2.2 HOW ERATOSTHENES DETERMINED THE RADIUS OF THE EARTH

Eratosthenes approached the problem by assuming the Sun was so far away that all 'sunbeams' reaching the Earth were, in effect, parallel. (We shall examine this assumption in detail in the TV programme associated with this Unit.) He then simultaneously compared the direction of the vertical at two different locations on the Earth's surface, with the direction of the parallel beams of sunlight at that instant (Figure 6). But how did Eratosthenes make *simultaneous* measurements at two different places on the Earth's surface? If the Greeks had had reliable clocks that could be synchronized and then transported, he could have asked an assistant to make the measurement at one of the locations at some previously agreed instant of time (as shown by one of the clocks), while he made the measurement at the other location at the same instant of time (as shown by the second clock). But the Greeks had no such clocks. So Eratosthenes solved the problem of synchronization by carrying out his measurements at noon (i.e. the time when the Sun was highest in the sky) at two places lying on what we should now call the same line of longitude.

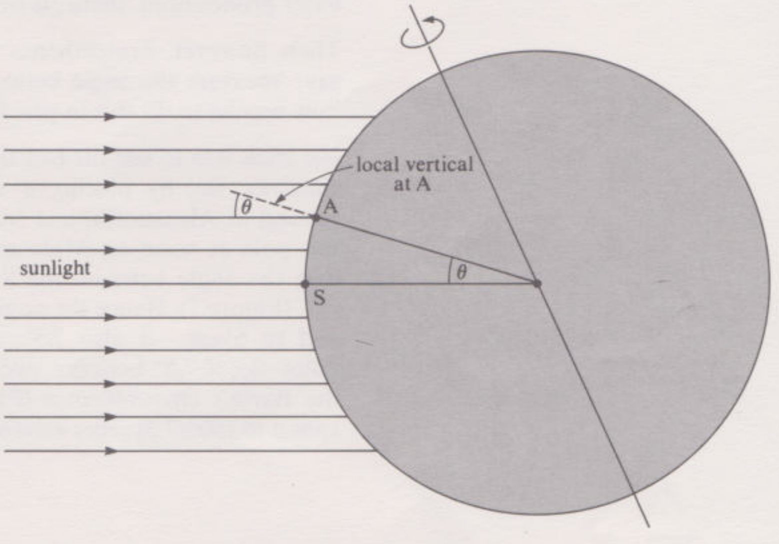


FIGURE 6 The parallel beams of sunlight provide a *reference* direction for all points facing the Sun. The direction of the local vertical (i.e. the direction in which a plumb-line would hang at that locality) is, by definition, the direction determined by a line passing through that point on the Earth's surface, and the centre of the Earth. In other words, the local vertical lies in the same direction as the Earth's radius at that location (plumb-lines point towards the centre of a spherical Earth).

□ Why did he choose two locations lying on the same line of longitude?

■ A line of longitude is a line drawn around the Earth in a north–south direction and passing through the two Poles. Consequently, all points on this line experience noon (i.e. that time of day when the Earth's rotation brings the Sun to its culmination point) at the same instant in time.

The two locations that Eratosthenes chose were Alexandria (where he worked) and Syene—now called Aswan—almost exactly 500 miles due south of Alexandria (A and S in Figure 6). It is easy for us to quote this distance nowadays, but Eratosthenes must have found the measurement very difficult to make. We are not sure exactly how he solved the problem—the various documented accounts are in disagreement on this point. Whatever technique he did actually use, it was fortunate that the terrain between Alexandria and Syene was sufficiently flat to allow the measurement to be made at all.

So what observations did Eratosthenes make at Alexandria and Syene, and how did his observations enable him to estimate the size of the Earth? Well, the observation at Syene was a very simple one. He knew from records kept at Syene that, at exactly noon on Midsummer's Day, sunlight falling on a

DEFINITION OF π

very deep well there reached the water surface and was reflected straight back up the well again. (What the records said was that the water in this particular well was only visible at noon on Midsummer's Day. We would now say that Syene lies on the Tropic of Cancer.)

What do you think Eratosthenes deduced from this fact?

Eratosthenes reasoned that the direction of the sunlight and the direction of the local vertical at Syene coincided at that particular instant (Figure 6). That is, at noon on Midsummer's Day, the Sun at Syene was exactly 'vertically' overhead. Hence the angle between the direction of the Sun's rays and the direction of the local vertical was zero degrees. (One degree is defined to be 1/360 of a rotation through a complete circle. In scientific texts one degree is usually written as 1°.)

Consequently, if Eratosthenes also measured the angle between the direction of the Sun's rays and the direction of the local vertical at *Alexandria* at noon on Midsummer's Day, he would, in practice, be measuring the angle between the Earth's radius to Syene and the Earth's radius to Alexandria. These two (equivalent) angles are both labelled θ in Figure 6. (θ, the Greek letter pronounced 'theta', is frequently used to denote angles.)

Then, however, Eratosthenes was faced with another problem. It is easy to say: 'measure the angle between the Sun's rays and the local vertical'. Yet how was he to do this in practice?

His trick was to use the fact that unless the Sun is directly overhead, it casts shadows. So, by placing a vertical pole, whose height he knew, in the ground at Alexandria, and by measuring the length of the shadow cast by this pole at noon on Midsummer's Day, Eratosthenes was able to deduce that the angle between the Sun's rays and the vertical at Alexandria was 7.5° (Figure 7). Hence the angle between the two Earth radii—to Alexandria and to Syene—is also 7.5°. Alexandria and Syene are separated by 500 miles. So, if 7.5° between two Earth radii correspond to 500 miles around the Earth's circumference (Figure 8), then 1° between radii would correspond to (500/7.5) miles around the circumference.

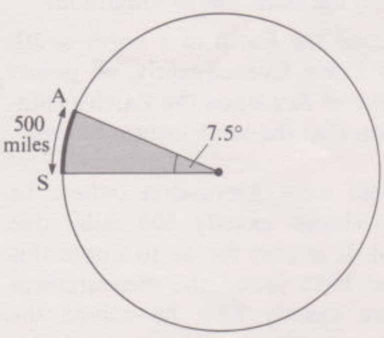


FIGURE 8 A is Alexandria and S is Syene. An angular separation of 7.5° between Earth radii corresponds to a distance of 500 miles around the circumference of the Earth. (Note that, to increase clarity, the size of the angle has been exaggerated in this diagram.)

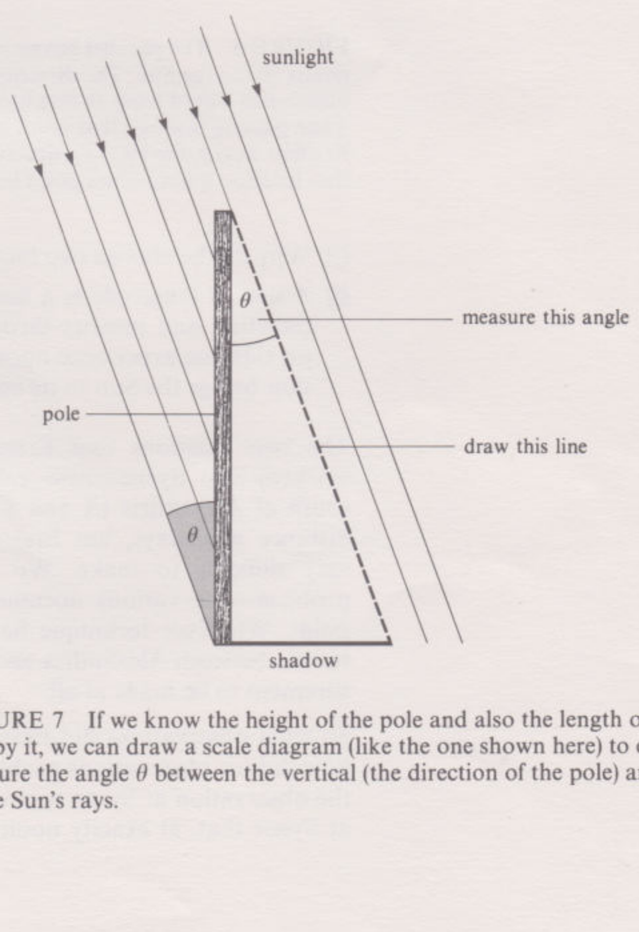
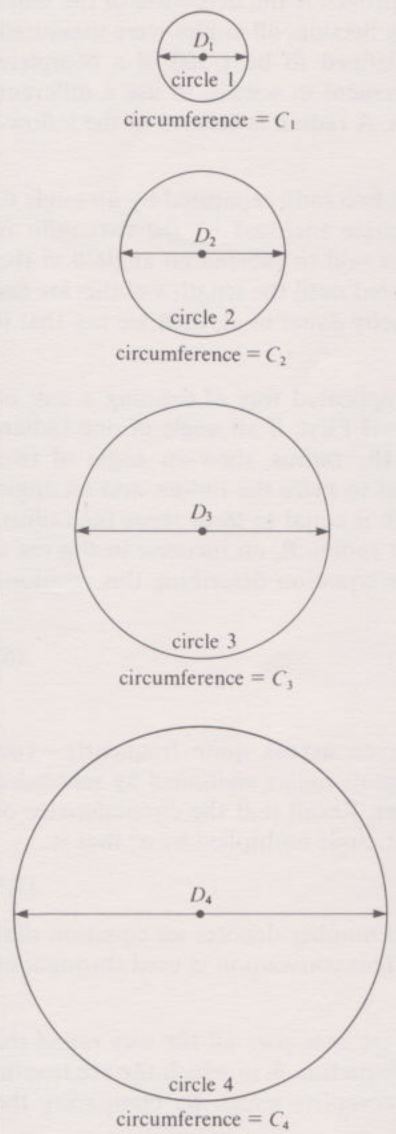


FIGURE 7 If we know the height of the pole and also the length of the shadow cast by it, we can draw a scale diagram (like the one shown here) to enable us to measure the angle θ between the vertical (the direction of the pole) and the direction of the Sun's rays.



Whatever the size of the circle, the ratio of circumference to diameter is always the same. That is

$$\frac{C_1}{D_1} = \frac{C_2}{D_2} = \frac{C_3}{D_3} = \frac{C_4}{D_4} = 3.14\ldots$$

(where this number has been rounded-off to three figures).

Nowadays we denote this constant ratio by the Greek letter π (pi).

So, the general rule is

$$\frac{\text{circumference}}{\text{diameter}} = \pi$$

or circumference = $\pi \times$ diameter where $\pi = 3.14$

FIGURE 9 The **definition of π**. You will also find π discussed in *MAFS 4*.

ITQ 3 What distance around the circumference, therefore, would 360° between Earth radii correspond to?

24,000

So, in this way, Eratosthenes was able to determine the approximate circumference of the Earth. Finding the radius was then a relatively easy step. The Greeks had long known that, for *any* circle, no matter how large or how small, the ratio between the circumference of that circle and its diameter is always the same (Figure 9). This fixed ratio is now denoted by the Greek letter π (pi).

$$\frac{\text{circumference}}{\text{diameter}} = \pi \tag{1}$$

This can be rewritten to give an expression for the circumference, by multiplying both sides of the equation by the diameter of the circle:

$$\frac{\text{circumference}}{\text{diameter}} \times \text{diameter} = \pi \times \text{diameter}$$

or

$$\text{circumference} = \pi \times \text{diameter} \tag{2}$$

It becomes very tedious having to write out the words circumference and diameter all the time. So it is common practice to represent these quantities by letters. If the circumference is denoted by C and the diameter by D, then Equation 2 becomes

$$C = \pi \times D$$

or

$$C = \pi D \tag{3}$$

(Notice that the multiplication sign between π and D is not necessary. If you see two symbols next to each other like this, the multiplication sign is implied.)

The final step is to realize that the diameter of a circle is just twice the radius. So if the radius is represented by the symbol R, then:

$$D = 2R \tag{4}$$

and Equation 3 can be written as

$$C = 2\pi R \tag{5}$$

(where we have simply replaced D by $2R$—the order in which quantities are multiplied together is immaterial).

You probably recognize Equation 5. Anyway, the important thing to notice is that if the circumference of the Earth is known, the radius of the Earth can be calculated.

$2\pi r$ 6.28

ITQ 4 In ITQ 3 you calculated Eratosthenes' value for the circumference of the Earth. If π is taken to be 3.14, what (according to Eratosthenes) is the *radius* of the Earth?

3821.65

ITQ 5 Given that 1 mile is approximately equal to 1.61 km, convert your answer to ITQ 4 into kilometres.

3.2.3 AN ALTERNATIVE PERSPECTIVE

There is an alternative way of analysing the problem described in Section 3.2.2, which is well worth taking a look at. To understand it, you need to learn some new mathematical techniques, but now that you have already seen the basic principles of Eratosthenes' method, and have worked out approximate values for the circumference and radius of the Earth, you are in a good position to follow through this alternative approach. Furthermore, the mathematics that you will learn en route will be needed later in this Unit.

RADIAN

SMALL-ANGLE APPROXIMATION

The starting point for this alternative approach is the definition of the units of angular measurement. In the preceding Section, all angles were measured in *degrees*, where one degree (1°) was defined to be 1/360 of a complete rotation. However, it is often more convenient in science to use a different unit of angular measurement—the *radian*. A radian is defined in the following way.

Figure 10 shows a circle of radius R with two radii separated by an angle θ. That part of the circumference of the circle enclosed by the two radii is known as an *arc* of the circle. (This arc is said to *subtend* an angle θ at the centre of the circle.) If the angle θ is adjusted until the length s of this arc (as measured *along* the circumference) is exactly equal to R, then we say that θ is defined to have a value of one **radian**.

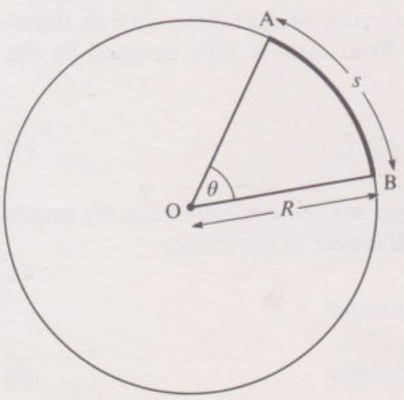


FIGURE 10 The angle θ encloses an arc of length s. Conversely, the arc s *subtends* an angle θ at the centre of the circle. The angle subtended when $s = R$ is defined to be 1 radian.

For a general angle of θ radians, $s = R\theta$ (see text).

At first sight this seems to be a very complicated way of defining a unit of angle, but there is method in the madness! First, if an angle of one radian means that the arc length is *equal* to the radius, then an angle of two radians means that the arc length is equal to *twice* the radius, and an angle of three radians means that the arc length is equal to *three times* the radius, and so on. In fact, for any circle of fixed radius R, an increase in θ gives a proportionate increase in arc length. The equation describing this relationship is:

$$s = R\theta \tag{6}$$

where θ is measured in radians.

Equation 6 is an equation you will come across quite frequently—you should memorize it. It says *arc length equals radius multiplied by subtended angle* (*in radians*). That is not all, however. Recall that the circumference of a circle is equal to twice the radius of that circle multiplied by π; that is:

$$C = 2\pi R \qquad (5)^*$$

(The asterisk to the right of the equation number denotes an equation that has already appeared earlier in the text. This convention is used throughout the Course.)

A circumference, however, is simply an arc *that goes all the way round the circle*. So Equation 5 is a special case of Equation 6, in which the arc length, and hence the angle, corresponds to a complete circle. By comparing the two Equations:

$$s = R\theta \qquad (6)^*$$

and $$C = 2\pi R \qquad (5)^*$$

$$= R \times 2\pi$$

we must conclude that the angle corresponding to a complete circle is 2π *radians*. Consequently:

$$\boxed{360^\circ = 2\pi \text{ radians}} \tag{7}$$

1 Radian

6.28 =

ITQ 6 A right angle is defined to be 90°. Express a right angle in radians.

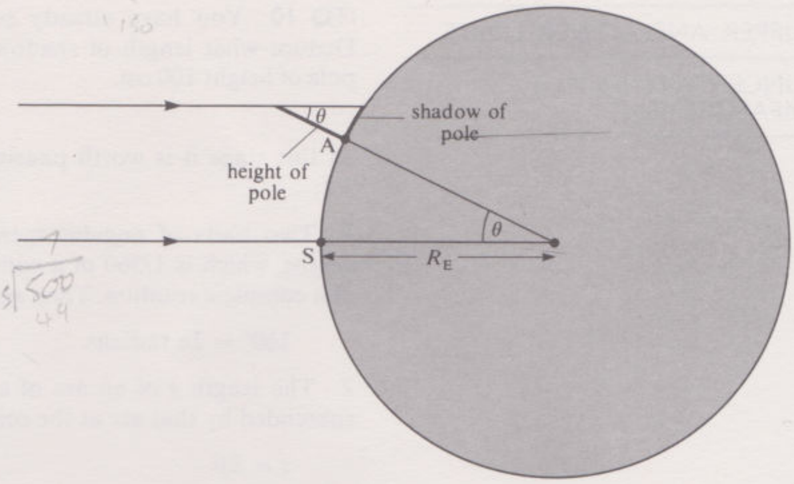
ITQ 7 Use Equation 7 to show that 1 radian ≈ 57.3°.

360 ÷ 6.283

ITQ 8 Use Equation 6 to deduce the *dimensions* of angle. Refer back to Section 2 if you've forgotten what is meant by the term 'dimensions'.

How can all this now be applied to Eratosthenes' data? Look at Figure 11. You have already seen that the angle between the direction of the Sun's rays and the vertical direction of the pole at Alexandria was the same as the angle between the two Earth radii to Alexandria and Syene respectively. So if this was 7.5° and the distance from Alexandria to Syene was 500 miles . . .

FIGURE 11 A is Alexandria and S is Syene. For clarity, the height and shadow of the pole at Alexandria and the angle θ have all been exaggerated in the diagram.



7.5
500
500 = R × 7.5
500/7.5
7.5)500
44
$\frac{S}{\theta} = R$

ITQ 9 Using Equation 6 this time, calculate R_E, the radius of the Earth, in miles. (The subscript E is used to remind you that the radius in question is the radius of the Earth. This type of subscript notation is very common in science.)

The only thing that is a bit dubious about this last calculation is the value of the angle θ. How was θ actually measured? It was suggested earlier that one way would be to measure the height of the vertical pole, and the length of the shadow cast by the pole, and then to draw a scaled-down diagram. The angle θ would then have to be measured with a protractor. But this diagram is not really necessary. If we don't mind making a slight approximation in our calculations, we can find the value of θ *directly* from the measurements of shadow length and pole height. We can say that:

$$\text{shadow length} \approx \text{pole height} \times \theta \text{ (in radians)} \tag{8}$$

Why? Look at Figure 12a. For the circle centred on the tip of the pole (i.e. B), using Equation 6 we can write:

$$\text{arc AD} = \text{radius BA} \times \theta$$

We are not, however, particularly interested in the length of the arc AD—we are much more concerned with the length of the shadow (AC) along the surface of the Earth. (In this notation 'line BA' means the line whose ends are defined by the points B and A.)

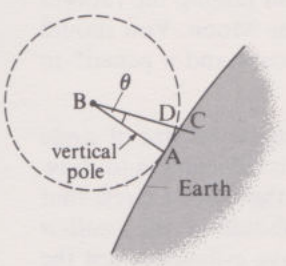
Look at Figure 12b, which is an enlargement of Figure 12a. On this scale you can see that the shadow AC is a straight line.* Furthermore, the length of the arc AD is almost indistinguishable from the shadow length AC. In fact, in order to make the difference between AC and AD visible, the angle θ in this Figure has been made greater than 15°, whereas Eratosthenes' angle was about 7.5°!

So, in summary, if θ is small, the arc AD can be treated as being approximately equal to the length of the shadow along the surface of the Earth. That is why Equation 8 is a valid approximation. In fact, when θ is small (less than about 15° or 0.26 radians), the curved arc length in Equation 6 can be approximated by a straight line. The smaller the angle, the better the approximation. (An angle of 15° leads to an inexactness of about 1%.) This approximation is sometimes called the **small-angle approximation**.

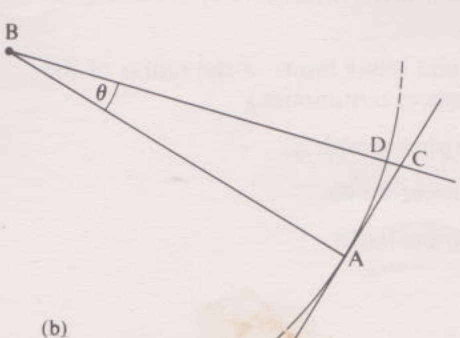
If we now rearrange Equation 8 to give an expression for θ, we find that, for small values of the angle,

$$\theta \approx \frac{\text{shadow length}}{\text{pole height}} \tag{9}$$

that is, the value of the angle (in radians) can be determined by calculating the ratio of shadow length to pole height.



(a)



(b)

FIGURE 12 If the angle θ is small enough, the length of the arc AD is approximately the same as the length of the straight line AC.

* The curvature of the Earth will have a negligible effect for these small distances.

UPPER AND LOWER LIMITS

UNCERTAINTIES IN A MEASUREMENT

ITQ 10 You have already seen that Eratosthenes found θ to be 7.5°. Deduce what length of shadow would have been cast at Alexandria by a pole of height 100 cm.

At this stage it is worth pausing to summarize the main points of Section 3.2:

1 Two units of angular measure are frequently used by scientists: the *degree*, which is 1/360 of a complete rotation and the *radian*, which is $1/2\pi$ of a complete rotation. They are related by the equation:

$$360° = 2\pi \text{ radians.}$$

2 The length s of an arc of a circle of radius R is related to the angle θ subtended by that arc at the centre of the circle, by the equation:

$$s = R\theta$$

where θ must be measured in radians.

3 When θ is small the curved arc length s can be taken as a straight line (the small-angle approximation).

3.3 THE RADIUS OF THE MOON

The Earth's nearest neighbour is the Moon. How does the size of the Moon compare with the size of the Earth? How *can* the size of the Moon be compared with the size of the Earth? The early astronomers knew that when the Earth passed between the Sun and the Moon a shadow of the Earth was thrown onto the Moon (Figure 13a). Figure 13b shows 20th century photographs of such a lunar eclipse. *If it is assumed that the shadow of the Earth on the Moon is the same size as the Earth itself*, then the ratio of the size of the Moon to the size of the Earth can be estimated from Figure 13b by completing the circle of the Earth's shadow, and finding the ratio of the radius of this circle to the radius of the circle of the Moon. You should now think about how you might use a pair of compasses and a pencil* to complete the circle of the Earth's shadow.

You will probably have some difficulty deciding exactly what size of circle best fits the arc of the shadow. The best way out of this difficulty is to draw not just one circle, but *two*. The first circle should be the *biggest* circle that you feel could be fitted to the arc; the second circle should be the *smallest* circle that could fit the arc. By doing this you will have estimated, not the *exact* size of the Earth's shadow, but the **upper and lower limits** to its possible size. You have found the size of the shadow to within certain tolerances. You will find that this is something you often have to do in science, namely to estimate the *range* of possible values that your measurement could cover. No measurement is ever exact—there will always be some **uncertainties** associated with it. So if you can say what the limits of these uncertainties are, other people do at least know what sort of credibility to give your measurement.

What values do you get for the upper and lower limits of the radius of the Earth's shadow? (Make the measurements in centimetres.)

The radius of the Earth's shadow in the photograph is:

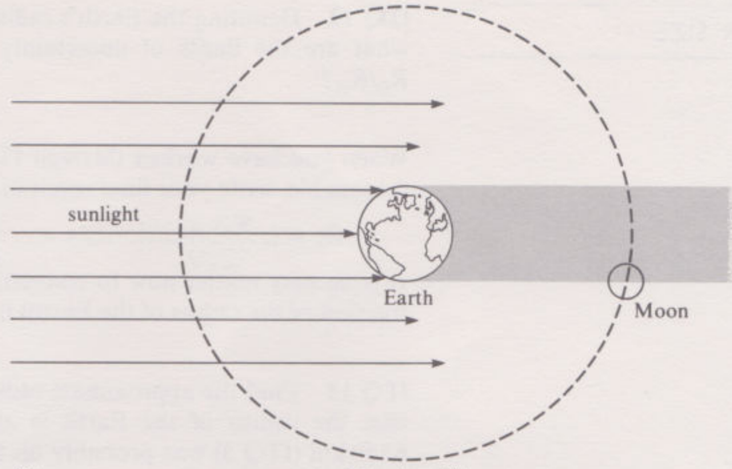
less than ....8.4........ cm (upper limit)

more than ....8.2........ cm (lower limit)

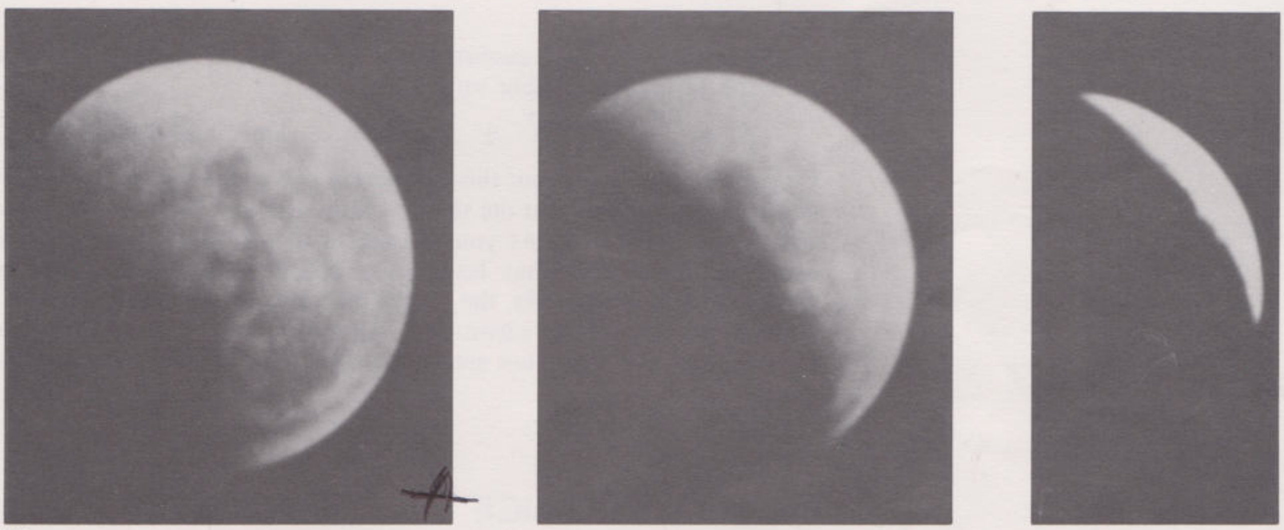
* If you are at all uncertain about how to use a pair of compasses to draw a circle, you should refer to *Into Science*, Module 9.

FIGURE 13 (a) The Earth, lying between the Sun and the Moon, casts a shadow on the Moon's surface. (As the TV programme shows, the way the Earth's shadow has been drawn here is not exactly correct, but this representation is adequate for the present argument.)

(b) Photographs showing three partial phases of the lunar eclipse of 2 May 1920. Notice that the less of the Earth's shadow you have, the more difficult it is to estimate R_E. On the other hand, the more of the Earth's shadow you can see, the more difficult it is to measure R_M.



(a)


(b)

ITQ 11 What, therefore, would you say is the *best estimate* you can give for the radius of the Earth's shadow in the photograph?

You will need to use this value again, so when you have completed ITQ 11 (and checked your answer against the one at the back), write your estimate in the space below.

Radius of Earth's shadow in the photograph = ...83... ± ...0,5... cm

The next measurement you need is that of the radius of the Moon (as shown in the photograph). Again make this measurement in centimetres:

Radius of the Moon in the photograph = ...2.9... cm

□ What do you estimate are the upper and lower limits for this measurement?

■ In this case, you probably felt that you could measure the radius of the Moon (in the photograph) to within about ±0.1 cm. This is a much smaller uncertainty than that involved in the measurement of the radius of the Earth.

So (given the assumption stated on p. 20) you can now say what the ratio is between the radius of the Earth and the radius of the Moon.

The radius of the Earth is ...3... times bigger than the radius of the Moon; that is,

radius of the Earth = ...2.93... Moon radii

ANGULAR SIZE

ITQ 12 Denoting the Earth's radius by R_E and the Moon's radius by R_M, what are the limits of uncertainty involved in your value for the ratio R_E/R_M?

When you have worked through ITQ 12, and checked that your answer is reasonable, write your final result in the space below:

$$R_E = (\frac{8.3}{2.7} \pm \frac{0.5}{0.1})R_M \quad 3.0 \pm 0.3$$

It is an easy matter now to convert the radius of the Moon (expressed as a fraction of the radius of the Earth) into a measurement in kilometres.

ITQ 13 Find the approximate radius of the Moon in kilometres, assuming that the radius of the Earth is about 6 200 km. (Eratosthenes' value of 6 150 km (ITQ 5) was probably up to 5% out. So, as we only want approximate values here, it is perfectly all right to work with the 'rounded-up' figure of 6 200 km.)

Again, write your final answer in the space below; don't forget to estimate the uncertainty in your value.

$$R_M = \frac{1 \times 6200}{3} \pm 2066 \text{ km} \quad 2066 \pm 207\text{km}$$

One word of caution about this calculation of R_M: *throughout Section 3* you make the assumption that the shadow of the Earth at the Moon is the same size as the Earth itself. As you will see in the TV programme, this is an *unjustified assumption* that leads to a considerable error in the value obtained for R_M. However, the programme also shows how to correct this error, so be prepared to adjust your value of R_M after viewing. (The essential details of this correction are summarized in the TV Notes, at the end of the text.)

3.4 THE DISTANCE TO THE MOON

3.4.1 ECLIPSING THE MOON

Now we know the approximate radii of both the Earth and the Moon. The next measurement we want to make is the *distance between* the Earth and the Moon. Apart from the Sun, the Moon appears as the largest body in the sky. But just because it *appears* to be the largest body, it does not necessarily follow that it *is* the largest body. Indeed, you probably already have a suspicion that the Moon appears larger than the stars because it is nearer than the stars! Apparent size must obviously depend not only on the real size of the object, but also on the distance of a object from you, the observer. The important point to realize is that the apparent size of the Moon is determined by the angle it subtends at your eye.

What actually determines the angle? Look at Figure 14. The **angular size** of the Moon in this diagram (i.e. θ_M) is clearly dependent both on the diameter of the Moon (D_M) and on the distance of the Moon from the observer on Earth (i.e. L_M). Once again, we can use the relation: arc length equals radius multiplied by subtended angle in radians. In this case the centre of the circle is point O in Figure 14, so the radius is L_M, the arc length is D_M and the subtended angle is θ_M. Admittedly, this is an approximation; but if θ_M is small, the error made by approximating the diameter D_M to an arc is very small. Remember that, even for θ as large as 15° (about 0.26 radians) the curved arc and the straight-line approximation to the curved arc differ only by about 1%. In the case of the Moon viewed from the Earth, θ_M is less than 1° so it is a valid approximation to write

$$D_M = L_M \theta_M \tag{10}$$

Rearranging Equation 10 to find an expression for θ_M, we have

$$\theta_M = \frac{D_M}{L_M} \tag{11}$$

$y = mx + c$.
gradient
intercept upon y axis.

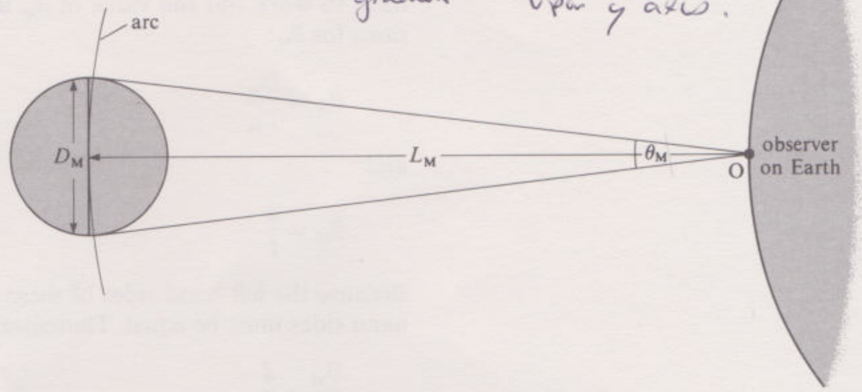


FIGURE 14 If you imagine a circle, centred on O, and passing through the centre of the Moon, the arc corresponding to the angle θ_M will be almost the same length as the diameter of the Moon D_M, provided that θ_M is not too large. Thus, we can write

$$D_M \approx \text{arc} = L_M \theta_M$$

(The subscript M is used to indicate that we are talking about those particular values of D, L and θ that are relevant to the Moon.)

You can see from Equation 11 that the angular size of the Moon is determined by the ratio of the Moon's diameter to the distance of the Moon from the Earth. In Section 3.3 you found the radius of the Moon. Use your value of R_M from Section 3.3 to work out the diameter of the Moon, D_M.*

$D_M =$ ...4132..... ± ......414...... km

If you could measure θ_M, you could deduce the distance between the Moon and the Earth. The question is how do you measure θ_M?

□ Look at Figure 15. This should give you a clue as to how to measure θ_M. Can you suggest a way?

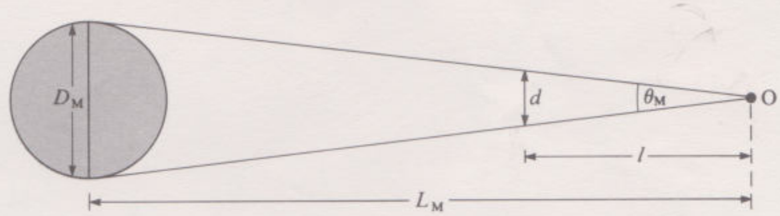


FIGURE 15 The line d and the diameter of the Moon D_M both subtend the same angle θ_M at O, i.e. they both have the same angular size. (The two triangles formed in this way have the same angles. Triangles like this, which have the same *shape*, but differ by a scaling factor, are called *similar* triangles.)

■ θ_M is given by D_M/L_M. But the small-angle approximation can also be applied to the 'arc' d, so that

$$\theta_M = \frac{d}{l} \qquad (12)$$

Thus, if you were to position some object of known diameter d (say) a distance l away from your eye, such that its *apparent* size (i.e. its angular size θ_M) was the same as that of the Moon, you could immediately say that θ_M, the angular size of the Moon, would be given by d/l. Thus, if l can be measured, you can find θ_M (since d is known).

ITQ 14 Your object of diameter d, when placed a distance l away from your eye, must look about the same apparent size as the (full) Moon. Given the hint that the angular size of the Moon is less than 1° (though no information as to how *much* less!), work out what *minimum* diameter an object must have if you want to use it to eclipse the Moon from a distance l of 1 metre.

The only question that remains to be answered is: how could you check that your object of diameter d really is subtending the same angle at your eye as the full Moon? Well, as you'll probably have already guessed by

* The ratio of D_M to the uncertainty in D_M must be the same as the ratio of R_M to the uncertainty in R_M. For example, if the radius of a circle is measured as (2.0 ± 0.1) m, then the diameter of that circle would be quoted as (4.0 ± 0.2) m.

now, the trick is *just* to eclipse the Moon with your object. You then have exactly the arrangement shown in Figure 15. In fact, you don't actually have to work out the value of θ_M in order to find L_M. We have two equations for θ_M:

$$\theta_M = \frac{D_M}{L_M} \tag{11}^*$$

and

$$\theta_M = \frac{d}{l} \tag{12}^*$$

Because the left-hand sides of these two equations are the same, their right-hand sides must be equal. Therefore:

$$\frac{D_M}{L_M} = \frac{d}{l} \tag{13}$$

Or, rearranging to find an expression for L_M, we have:

$$L_M = \frac{D_M l}{d} \tag{14}$$

Note that all the quantities on the right-hand side of this equation are either known or can be measured.

reverse

$$L_M = \frac{D_M}{1} \div \frac{d}{l} \rightarrow \frac{D_M \times l}{1 \times d}$$

$$\frac{d}{l} = \frac{D_M}{L_M} \qquad d = \frac{L_M}{l\,D_M}$$

EXPERIMENT

DETERMINING THE DISTANCE BETWEEN THE MOON AND THE EARTH

TIME

The practical part of this experiment takes about 30 minutes.

NON-KIT ITEMS

piece of dowelling or a broom handle (or some other type of straight rod) at least 1.2 metres long

tape measure or rule at least 1.5 metres long

Blu-Tack or plasticine

KIT ITEMS

Part 1

four plastic discs

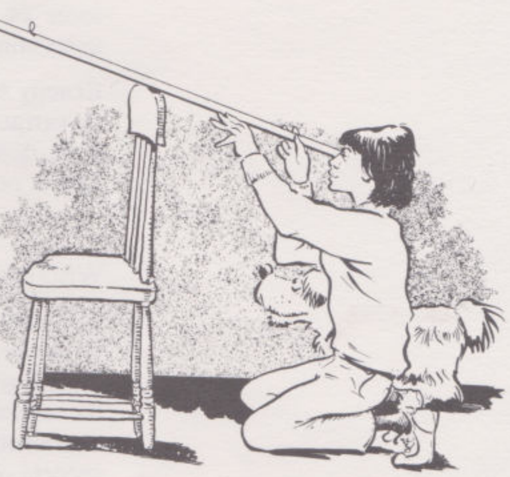


FIGURE 16 One possible arrangement for eclipsing the Moon.

SETTING UP THE EXPERIMENT

You now have all the information you need to enable you to find L_M, the distance between the Earth and the Moon. What you need to do is to devise for yourself a simple experimental arrangement that enables you to eclipse the Moon. Figure 16 should set you thinking along the right lines, but you can probably improve on the arrangement illustrated there. After all, it's *your* experiment, so if you think you may be able to get a more accurate estimate of L_M by modifying either the equipment or the technique, then feel free to improvise your own method.

Ideally you require a clear sky and a full Moon to do the experiment properly, but even without these ideal conditions you should still be able to get some kind of result. For instance, if you are careful about matching the curvature of the disc to the curvature of the Moon, you should (with a bit of patience) be able to make the measurements on much less than a full Moon. And although there is not much you can do about cloudy nights, you can, at least, practise your eclipsing techniques on household objects (a standard light bulb at 10 metres is suitable for a test run). You will then be well prepared to take full advantage of the first available cloud-free night.

Tips:

1 For various reasons that are not appropriate to go into at this stage, you will get a more accurate result if you do the experiment in daylight or twilight, rather than on a *dark* night. Alternatively, you could try to work from the window of a well-lit room, rather than doing the experiment outside in the dark.

2 You will find it virtually impossible to block out all the light from the Moon—there will always be some haze around your eclipsing disc. Try instead to match the *curvature* of the disc to the *curvature* of the Moon.

3 You won't find it very easy to decide exactly where the optimum eclipse position is. You faced a similar problem to this in Section 3.3, when you were trying to estimate the radius of the Earth's shadow in the lunar eclipse photograph. There, you found the likely limits, both upper and lower, for this radius. You should do the same sort of thing here; try to estimate the furthest and closest possible eclipse positions, and take the average of these. Remember you should *always* try to assess the uncertainties in every measurement you make.

EXPERIMENT CONTINUED

RECORDING THE RESULTS

The quantities you measure are those on the right-hand side of Equation 14, i.e. the distance l and the disc diameter d. You have several discs, so you need to consider which ones will give you the most accurate answers. You'll probably want at least to *try* them all, and to repeat some of the measurements several times. Don't forget to assess the uncertainties in your values of d as well as in the values of l.

Exactly *how* you record your results is up to you, but you must be able to extract the information at a later stage. The important thing is to write down in your Notebook every aspect of what you did so that you have a permanent record. Don't rely on memory!

WRITING UP A REPORT OF THE EXPERIMENT

You will be asked to 'write-up' this experiment for your first TMA. However, you should *not* attempt to produce the report at this stage. Unit 3 gives you a chance for further practice at such skills as estimating the overall uncertainty when two uncertain quantities (e.g. l and d) are combined. Unit 4 also gives detailed advice about how to construct reports of practical work, with specific reference to this experiment. *You should therefore defer writing up Question 1 of TMA 01 until you have read Unit 4.* Just make sure for now that you have recorded in your Notebook the upper and lower limits to your disc diameter(s) and to the corresponding eye-to-disc distance(s).

But don't be deterred from making a quick calculation, just to see that your data do give a sensible value for L_M: simply substitute your average results for l and d into Equation 14. One final point in connection with that equation: remember that you have to revise your value of R_M (the radius of the Moon) after watching the TV programme. You should, of course, eventually use this revised value in Equation 14, to calculate L_M—don't use the value of D_M you calculated on p. 23.

3.5 THE DISTANCE TO THE SUN

One of the first accurate estimates of the distance of the Moon from the Earth was made by another early Greek 'astronomer', Aristarchus (about 240 BC). And unbelievable though it seems, using a technique very similar to the one you have just used, he actually obtained a result that was within a few per cent of our present-day value! When Aristarchus tried his hand at estimating the distance from the Earth to the *Sun*, however, he got a value that, when compared with the present-day one, was a factor of 20 times too small! Why was he so wrong?

The simple answer to this is that (even today) the distance to the Sun is much harder to estimate than the distance to the Moon, because the Sun is much further away. Another reason for Aristarchus's error being so large was that he used an *indirect* technique. This, though ingenious, had the defect that a small error in the quantity he actually measured led to an enormous error in the quantity he was ultimately trying to find. Follow Aristarchus's reasoning through, and see if you can spot where this huge error creeps in.

Aristarchus argued as follows. The Moon goes through various phases—new Moon, half Moon, full Moon, etc.—as the position of the Moon changes relative to the positions of the Earth and the Sun. (Recall Section 3.3 of Unit 1.) Thus when the Moon appears exactly as a half Moon (i.e. first or last quarter), the sunlight must be striking the Moon at right angles (i.e. 90° or $\pi/2$ radians) to the line of sight of the observer watching the Moon. This situation is illustrated in Figure 17. At this particular time, the angle between the direction of the Moon and the direction of the Sun (labelled ϕ (Greek letter phi) in Figure 17) is measured. This angle is

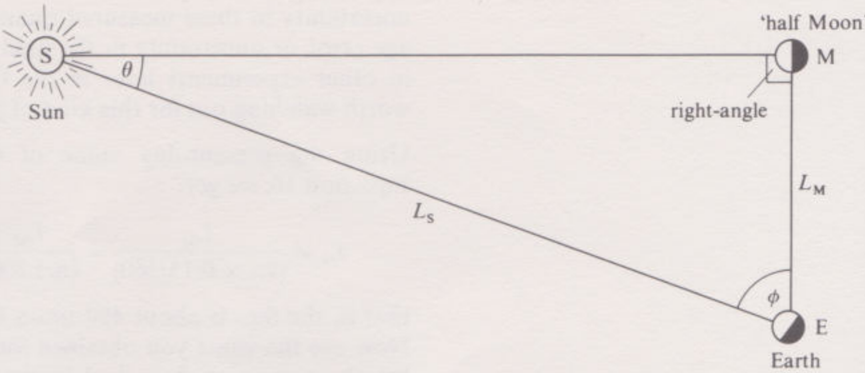


FIGURE 17 Aristarchus estimated L_S, the distance from the Earth to the Sun, by measuring the angle between the Moon and the Sun (the angle labelled ϕ in the diagram) at the moment when the Moon appeared to be exactly a 'half Moon'. He then deduced θ. Since θ was small, he used the equation $L_M = L_S\theta$ (which is just arc $= R\theta$) to find L_S. (The angle θ has been exaggerated in this diagram to make it clearer.) Aristarchus estimated that $\theta \approx 3°$.

almost—but not quite—a right angle (though it is much less than this in the Figure, where the scales are very distorted!).

The Greek mathematicians knew that *the sum of all angles in a triangle is equal to 180°*. So, since the angle at M is a right angle, it follows that:

$$\theta + \phi + 90° = 180°$$

$$\text{so} \qquad \theta = 90° - \phi \tag{15}$$

Aristarchus measured ϕ to be about 87°, so he deduced that $\theta \approx 3°$. Now 3° is a small angle, so we can make use of the small-angle approximation for the equation: arc $= R\theta$. Thus we can write:

$$\text{arc} \approx L_M$$

$$R \approx L_S$$

Therefore

$$L_M = L_S\theta$$

$$\text{or} \qquad L_S = \frac{L_M}{\theta} \tag{16}$$

ITQ 15 Calculate (using Aristarchus's value of $\theta = 3°$) how many 'Moon-orbit distances' the Earth is away from the Sun. That is, find the ratio ratio L_S/L_M.

The present-day values for the Earth–Sun and Earth–Moon distances (listed on the back cover) give a ratio L_S/L_M of roughly 400. Aristarchus's result is 20 times too small. Why?

The currently accepted value for the angle ϕ is approximately 89.85°. Aristarchus measured this angle as 87°. If we compare Aristarchus's value of ϕ with the present-day one, it is clear (in retrospect) that his measurement was in error by about $(89.85 - 87)° = 2.85°$. An error in measurement of 2.85° in 87° can be expressed in the form of a percentage as:

$$\frac{2.85}{87} \times 100\% \approx 3\%$$

At first sight, this seems quite a small discrepancy, but let us analyse things in a little more detail.

The problem with Aristarchus's method was that he did not use the value of ϕ *directly* to find L_S. As Equations 15 and 16 show, the angle used in the calculations is not ϕ, but θ, which is $(90 - \phi)°$. His value of ϕ was thus $(90 - 87)° = 3°$, when it should have been $(90 - 89.85)° = 0.15°$. Even though Aristarchus's determination of ϕ was only 3% out, his value of θ was a factor of 20 (i.e. 3°/0.15°) too large, and it was this inaccuracy that caused his result for L_S to be a factor of 20 too small.

There's a moral here. Whenever a calculation involves taking the difference between two quantities that are nearly equal, a small percentage error or

uncertainty in these measured quantities can produce a very large percentage error or uncertainty in the final result. You will meet similar situations in other experiments later in the Course; in practical work, it is always worth watching out for this kind of problem.

Using the present-day value of $\theta = 0.15° = (2\pi \times 0.15/360)$ radians in Equation 16, we get:

$$L_S = \frac{L_M}{(2\pi \times 0.15/360)} = \frac{L_M}{(\pi/1\,200)} \approx 400\ L_M \tag{17}$$

that is, the Sun is about 400 times farther away from us than is the Moon. Now use the value you obtained for the distance to the Moon (after applying the correction described in the TV programme) and substitute it into Equation 17 to calculate the distance from the Earth to the Sun:

$L_S = \ldots\ldots\ldots\ldots\ldots\ldots$ metres

3.6 THE RADIUS OF THE SUN

How big do the Sun and the Moon look to you? Or, to ask the same question more scientifically: How do the apparent sizes of the Sun and Moon compare? Think about this question for a minute. The interesting observation is that the Sun and Moon *look roughly the same size*. However, it's difficult to be precise about this, because the Sun is so much brighter than the Moon. Is there any independent evidence to support this impression that the Sun and Moon have the same angular size?

Actually there is. You've probably seen, at some time or another, photographs of solar eclipses. You've perhaps actually observed annular eclipses (Figure 18a), or even total eclipses (Figure 18b). These provide direct evidence that the Sun and Moon have approximately the same angular size. The Earth–Sun–Moon geometry is shown in Figure 19.

Note: **Under no circumstances should you try to eclipse the Sun with your plastic discs. You will damage your eyes.**

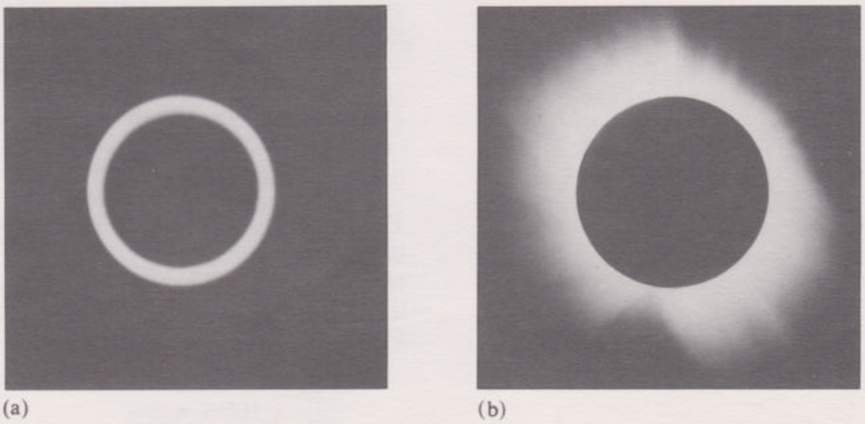


FIGURE 18 (a) A photograph of an *annular* solar eclipse. In this type of eclipse the Moon does not quite block out all of the Sun.

(b) A photograph of a *total* solar eclipse. In this case the basic disc of the Sun is completely obscured by the Moon so allowing observation of the flares in the Sun's 'outer atmosphere'.

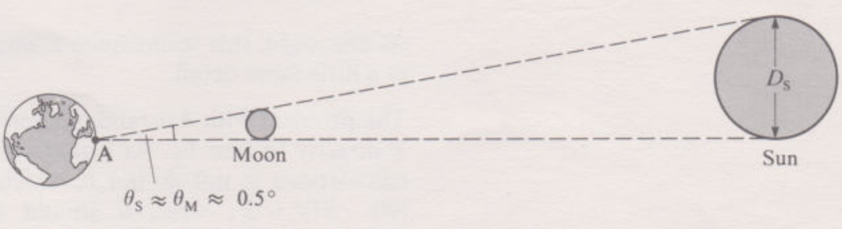


FIGURE 19 The Sun and the Moon have approximately the same angular size—they both subtend an angle of about 0.5° at the Earth. This diagram, which is not to scale, shows how a total eclipse is produced at the point A on the Earth's surface. An annular eclipse (like that shown in Figure 18a) is seen when the Moon is slightly further away from the Earth. Two types of eclipse occur because the Moon's orbital radius varies slightly.

Figure 19 shows basically the same arrangement as the one you used in the experiment (Section 3.4) to find the distance to the Moon. The only difference is that here the Moon is eclipsing the Sun, whereas in your experiment a plastic disc was eclipsing the Moon. Nevertheless, the analysis must be identical, that is

$$\theta_S = \theta_M$$

where $\theta_S = \dfrac{D_S}{L_S}$ and $\theta_M = \dfrac{D_M}{L_M}$

so that

$$\frac{D_S}{L_S} = \frac{D_M}{L_M} \tag{18}$$

Or, multiplying both sides of Equation 18 by L_S:

$$D_S = D_M \frac{L_S}{L_M} \tag{19}$$

You can now use your value for D_M from Section 3.3 (corrected in the light of the TV programme) to estimate D_S. Remember that Equation 17 shows that the approximate value of the ratio L_S/L_M is 400.

D_S = ... metres

Therefore

radius of the Sun, $R_S = D_S/2$ = ... metres

3.7 BRINGING THE RESULTS TOGETHER

You are now in a position to be able to summarize the sizes and distances involved in the Earth–Sun–Moon system by completing Table 4, using your own data whenever possible. Standardize on units by quoting all your results in metres. (Take care to give your values the correct powers of ten.) As your starting point in this Table, you should use the presently accepted value for R_E (the radius of the Earth) of 6.38×10^6 metres. The values you write down should, of course, take account of the correction to R_M discussed in the TV programme and in the TV Notes at the end of this Unit.

In the right-hand column of Table 4, you should make some comment on the accuracy of the result you give. If possible, write down the uncertainty in numerical form, but if this is not possible (because the result is a combination of your data with data taken from the text, for instance) you should try to give some idea of the reliability of the result in words.

ITQ 16 How many Earth radii is the Sun away from the Earth?

TABLE 4 Summary of results

Measurement	Comment on accuracy
R_E, radius of Earth = 6.38×10^6 metres	
R_M, radius of Moon = ... metres	
R_S, radius of Sun = ... metres	
L_M, distance of the Moon from the Earth = ... metres	
L_S, distance of the Sun from the Earth = ... metres	

TANGENT TO A CIRCLE

SUMMARY OF SECTION 3

In this Section, using just a pole in the ground, some photos of the Moon, a length of dowelling, a few plastic discs, and a lot of ingenuity, you have been able to estimate the size of the Earth, the size of the Moon, and the distance between the Earth and the Moon. Quite impressive! Equally valuable, however, is the fact that, in taking and analysing these measurements, you have made use of a number of mathematical ideas and scientific skills, the most important of which are listed below.

1 The circumference C of a circle is related to its radius R (and its diameter D) by the equation $C = 2\pi R = \pi D$.

2 There are two units of angular measure in common usage: radians and degrees (°).

$$2\pi \text{ radians} = 360° = 1 \text{ complete circle}$$

Hence $\quad 1 \text{ radian} = (360/2\pi)°$

and $\quad 1° = (2\pi/360) \text{ radians}$

3 In a circle,

$$\text{arc length} = \text{radius} \times \text{subtended angle (in radians)}$$

i.e. $\quad \text{arc} = R\theta$

4 If θ, in the equation $\text{arc} = R\theta$, is small (i.e. less than about 0.26 radians or 15°), then the alternative equation $s = R\theta$, where s is the straight-line approximation to the curved arc, is also true to an accuracy of better than about 1%. This is known as the small-angle approximation.

5 A 'best estimate' for the value of a measurement can generally be found by taking the average of the upper and lower limits of that measurement.

6 Measured quantities should be expressed in the form: best estimate plus or minus the uncertainty in the measurement, and the units must always be given,

i.e. $\quad \text{quantity} = (X \pm x) \text{ units}$

where X and x are numerical values.

4 THE PLANETS

4.1 COPERNICUS'S CONTRIBUTION

Although the Greek astronomers of Alexandria were able to make quite reasonable estimates of the dimensions of the Earth–Sun–Moon system, they certainly didn't know the distances to the planets or to the stars. (The Greeks did realize that there was a difference between the planets and the stars: the planets 'wandered' about relative to the constellations.) The best they could do was to presume that the planets were further away than the Moon, and the stars further away than the Sun and planets. It was not until the beginning of the 16th century when Nicolas Copernicus (1473–1543) developed his theory of planetary orbits—with a stationary Sun at the centre of things (Figure 20)—that it became possible to estimate the *relative* distances to the planets. But once the idea had been mooted of a Sun-centred system, with the planets travelling around the Sun in circular orbits, it became possible—admittedly with some fairly complicated reasoning—to begin calculating the relative radii of these orbits.

For instance, Copernicus deduced the ratio of the radius of the Earth's orbit to that of Venus in the following way. (Recall from Unit 1 that Venus's orbit is closer to the Sun than is the Earth's.) Venus's orbit lies in almost the same plane as the Earth's so that, seen from the Earth, Venus seems merely to swing to and fro, relative to the Sun, first to the left then to the right of it, sometimes passing in front and sometimes behind (Figures 21a and 21b). As you can see from the two diagrams in Figure 21, the extreme right-hand position of Venus's apparent oscillation (i.e. position B) is reached when the line of sight from Earth to Venus *just touches* the circu-

FIGURE 20 The Copernican system of planets with the Sun at its centre. Aristarchus had actually suggested a Sun-centred system back in the third century BC. Unfortunately, he was ahead of his time—tradition and 'logic' were against him. Furthermore, he was not able to make any measurements to support his 'peculiar' theory.

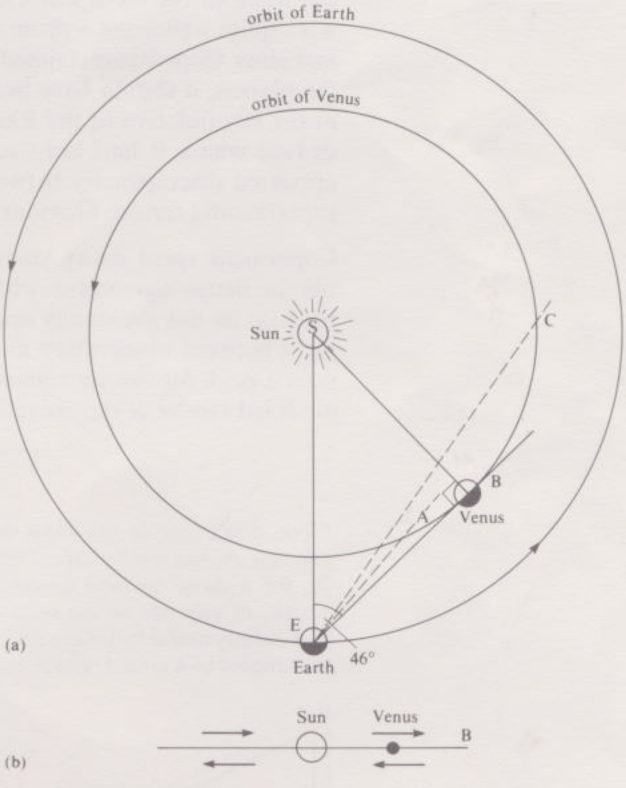


FIGURE 21 (a) The planet Venus orbits the Sun in more or less the same plane as the Earth. Its orbital radius is smaller than the Earth's.

(b) Venus's orbit, as seen from the Earth, appears to oscillate backwards and forwards across the Sun in a straight line.

lar orbit that Venus is assumed to be following. This line EB, which just touches the circle that is Venus's orbit, is said to be a **tangent** to that circle. You can see that a line of sight to a point earlier in the orbit (e.g. EA), or a line of sight to a point later in the orbit (e.g. EC), always corresponds to a smaller angle of deviation from the Sun's direction. The tangent to the circle corresponds to the angle of maximum deviation of the line of sight. Now the Greek mathematicians had shown long before that *a tangent to a circle is always at right angles to the radius of the circle that passes through the tangent's point of contact*. So in Figure 21a we can say that when Venus appears to be at its maximum angular distance from the Sun (i.e. at B in Figure 21b), then the angle at B (denoted as angle EBS) is 90°. If we were also to measure the angle at E (i.e. angle SEB) at this instant of time—which we could do by simple sighting from the Earth—then we could deduce the third angle in the triangle, angle BSE. Copernicus found the angle SEB to be 46°. He therefore deduced, using the fact that the sum of the angles in a triangle is 180°, that the angle BSE (the angle at S) must be 44° (46° + 44° + 90° = 180°).

SINE (SIN)

COSINE (COS)

TANGENT (TAN)

HYPOTENUSE

Once all the angles of the triangle are known, a scaled-down version of this triangle can be drawn. As you may remember from Figure 15, triangles that are identical apart from a scaling factor are called 'similar' triangles. If the Earth–Sun distance is drawn (arbitrarily) 100 mm long, then it turns out that the Venus–Sun distance must be about 72 mm long. (Try it, if you're not convinced.)* Hence the radius of Venus's orbit must be 0.72 times the radius of the Earth's orbit.

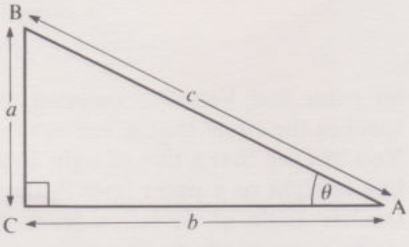
Unfortunately, for Copernicus to turn this ratio of orbital radii into an absolute value for the radius of Venus's orbit, he needed an accurate value for the radius of the Earth's orbit (i.e. the value of L_S from Section 3.5). We know this distance quite accurately nowadays, of course, but the best estimate Copernicus had was that determined by the early Greeks—and that value was a factor of 20 out. Consequently, Copernicus got the orbital radius of Venus wrong by this factor.

Needless to say, there was nothing special about Venus. Copernicus repeated similar calculations for the other planets and, although the final *scale* of his Solar System was wrong, he did manage to get the *ratios* between the radii of the planetary orbits roughly correct.

Notice, however, that implicit in Copernicus's calculations was the assumption that the orbits were perfectly circular. We now know that this was not a correct assumption. Because of this, the Copernican measurements did not quite fit the facts. For example, since the orbital periods of the planets were quite well known (from the many years' records of detailed sightings), and since Copernicus claimed to have calculated the relative orbital radii of the planets, it should have been possible to *predict* where a planet would be in the sky (relative to the Earth) at some specified time in the future, or to deduce where it had been some time in the past. It was here that there appeared discrepancies between predictions of the Copernican model and experimental results. Obviously, his model was not satisfactory.

Copernicus spent many years trying to modify his simple model so as to obtain better agreement with the observations. In a way, he succeeded, because he did eventually manage to produce a model in which the agreement between observation and theoretical prediction was very good. But he paid a price for this agreement: his model ceased to be simple! In truth, his modified model of the Solar System, with the Sun at the centre, was just as

* You don't *have* to draw this triangle to find the ratio of two of the sides. Instead you can use the mathematical techniques of *trigonometry*. (See *Into Science*, Module 10, for a more detailed discussion.) In any *right-angled triangle*, the ratio of the lengths of any two of the sides can be related to the angles of that triangle. Three particularly useful ratios are **sine, cosine** and **tangent** (this tangent is *nothing* to do with the tangent to a circle), which are defined thus for the angle θ:



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b}$$

where **hypotenuse** is the name given to the side facing the right angle, *opposite* denotes the side opposite angle θ, and *adjacent* is the name given to the shorter side adjacent to θ.

These quantities have been worked out for all angles between 0° and 90° and tabulated in trigonometrical tables, or programmed into pocket calculators.

So in Figure 21a,

$$\sin 46^\circ = \frac{SB}{SE} = \ldots\ldots\ldots\ldots\ldots\ldots ?$$

(*Answer:* 0.72)

complicated as the 'old-fashioned' geocentric (Earth-centred) models. It was perhaps this fact, more than anything else, that gave the champions of the geocentric system—mainly the philosophical and religious bodies of the time—confidence in the 'rightness' of their case.

4.2 TYCHO BRAHE'S TABLES

The Danish astronomer Tycho Brahe (1546–1601) adopted a quite different approach to that of Copernicus. Rather than invent a model and then try to refine it so as better to fit the available data, he decided to improve the quality and quantity of the data. So he determined to keep a record of observations of the positions of all the planets (five, plus the Earth, were then known) at regular periods throughout the year. In fact, this collection of data became his life's work; he recorded the planetary positions not just for one year, but for more than 20 years. And because of his painstaking development of more and more accurate sighting devices (Figure 22), many of these planetary positions were determined to an accuracy of better than $(1/60)^\circ$ and some to better than $(1/360)^\circ$! So Tycho Brahe left to posterity the most accurate and comprehensive catalogue of 'heavenly activities' that history had so far seen. It is no exaggeration to say that these details formed the basis of future developments in the theory of planetary motion. For one of the most challenging tests that any new theory had to pass was that it had to fit this wealth of data compiled by Brahe.

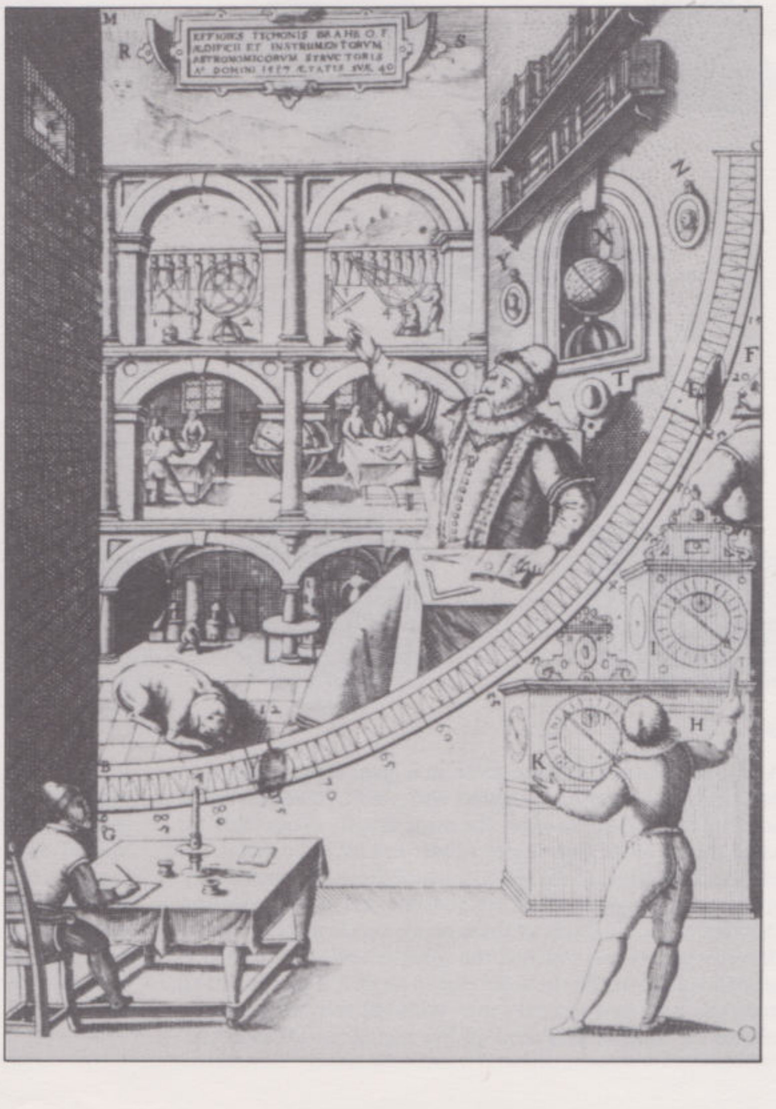


FIGURE 22 One of the most important sighting instruments in Tycho Brahe's observatory at Uraniborg in Denmark was this huge brass quadrant arc. The arc itself was securely fixed into a western wall, and a south-facing open window was located at the centre of the arc. The empty wall-space inside the arc was decorated with a mural showing Brahe, his dog and his laboratories. In this sketch (taken from Brahe's own book), an observer looking through a pin-hole at F (on the extreme right) is locating the position of a 'star' to an accuracy of better than $(1/60)^\circ$—in fact the scale of the instrument could be read to $(1/360)^\circ$.

ELLIPSE

KEPLER'S FIRST LAW

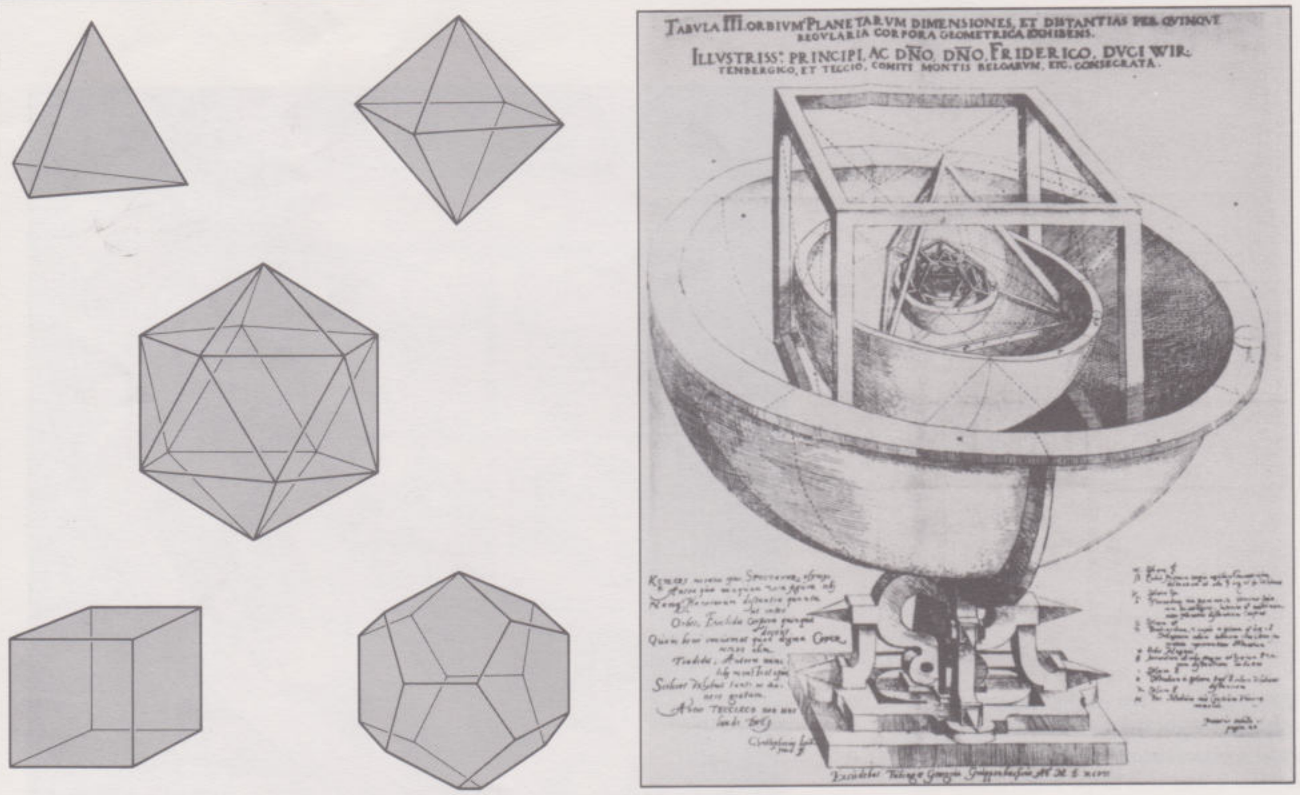
4.3 KEPLER'S SEARCH FOR REGULARITY

4.3.1 KEPLER THE 'PROBLEM SOLVER'

Johann Kepler (1571–1630), born in Germany, was a generation younger than Tycho Brahe. As astronomers, they were more or less exact opposites. Tycho was a brilliant experimenter and observer. His mechanical ingenuity, as witnessed by his development of numerous observational aids, seemed boundless. Kepler, on the other hand, was a solver of puzzles. He had the kind of mind that delighted in, and was fascinated by, the relationships between numbers, or sizes, or geometrical shapes. So, in some ways, Kepler was the obvious man to tackle the puzzle posed by Tycho Brahe's tables.

Kepler also had an almost mystical belief that there was some mathematical scheme underlying the planetary system. Why only six planets? Why were the planets' orbital radii in the ratio 8 : 15 : 20 : 30 : 115 : 195? (These were roughly the relative radii calculated, from Tycho's data, for the Copernican scheme of planets.) Kepler felt sure that there was some sequence to these numbers, and some mathematical explanation for there being only six planets. We know now, of course, that he was wrong. As you will see in Unit 3, the laws of gravitation allow a planet to orbit the Sun at *any* radial distance—so the *ratio* of the orbital radii can have no significance. Furthermore, we now know that there are more than six planets—we have added Uranus, Neptune and Pluto to the list. Nevertheless, after months of work, Kepler did come up with an explanation for the ratio of radii—an explanation based on the geometry of the five regular solids (Figure 23). He was so pleased with his explanation that he published it in a book, copies of which

FIGURE 23 The ratios of the orbital radii of the planets—the right answer . . . but the wrong reason. (a) The five regular solids, (b) Kepler's scheme of regular solids (taken from his book).



(a)

(b)

A *regular solid* is the name given to a geometrical solid that has all its faces identical, regular and plane. Hence a regular solid must have all its edges the same length, all its faces the same shape, all its face angles equal, and all its corners identical. Because of the limitations of three-dimensional space, only *five* such solids are possible (see (a)). Kepler had the idea of 'nesting' one of these solids inside a sphere (so that its corners just touched the sphere), and then nesting a second sphere inside the solid (again so that it just touched), and then nesting a second solid inside the second sphere, and so on, until he had used all five regular solids (see (b)).

By choosing the correct sequence of regular solids he managed to get close agreement between the ratios of the planets' orbital radii, and the ratios of the radii of the 'nested' spheres. Furthermore, since there are only five regular solids, it is only possible to have *six* 'nested' spheres, thus explaining why there were only six planets! **Kepler was wrong**. We now know that this arrangement between sphere radii and planetary orbital radii *was pure coincidence*. We now also know that there are *nine* planets, not six.

he sent to Tycho Brahe and the Italian scientist, Galileo. Both scientists were favourably impressed, and Tycho Brahe invited Kepler to go to Prague to work with him on observations of Mars, 'the difficult planet'. So it was that Kepler became acquainted with the detailed tables drawn up over the years by Tycho. Indeed, the tables were still unpublished when Tycho died, and it fell to Kepler to publish them for him posthumously.

4.3.2 KEPLER'S FIRST LAW

At the time of Tycho Brahe's death, Kepler was deeply involved in a detailed study of the orbit of Mars. Using the data accumulated by Brahe, he tried to fit Mars first into a circular, Sun-centred orbit and then into a circular orbit with the Sun off-centre. Neither worked. Eventually it became clear to him that he would have to plot out, point by point, an accurate and detailed picture of the real orbit of Mars. The problem was not a simple one. All the information was there in Tycho Brahe's tables, but in a scrambled form. The difficulty was that the data gave the apparent position of Mars as seen from a moving Earth.

However, Kepler persevered, and after much calculation he did determine the *shapes* of the orbits of the Earth and of Mars. The Earth's orbit was very nearly circular; indeed, the apparent deviation from circularity could perhaps have been attributable to experimental uncertainty. But Mars was quite different. The orbital path that he had plotted out for this planet was far from being circular; it was quite clearly 'oval' in shape. Kepler failed for several years to guess the exact form of this particular oval shape, but eventually he realized that it was an **ellipse** with the Sun at one focus (Figure 24). He then tested out the ellipse idea (again with the Sun at one focus) on the other known planets in the Solar System. It worked. We now know this result as **Kepler's first law.**

KEPLER'S FIRST LAW

The planets of the Solar System orbit the Sun along elliptical paths. The Sun is at one focus of the elliptical orbits.

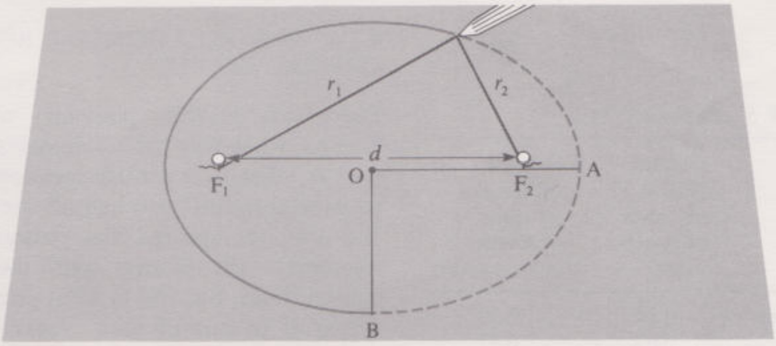


FIGURE 24 Defining an ellipse.

An ellipse is a very easy shape to draw if you have two drawing pins, a piece of string and a pencil. First, fasten the ends of the string to the 'pin-parts' of the two drawing pins. Now press the drawing pins into your drawing surface a distance d apart, where d is less than the length of the string. Take a sharp pencil, and with the tip, extend the string until it is taut. Now draw the curve which the pencil follows when it is moved in such a way as to keep the string taut. This curve is an ellipse; the pins are at the foci of the ellipse. An ellipse can be defined as that curve for which the sum of the distances from the two foci to any point on the curve, is constant, i.e. in the diagram, $r_1 + r_2 =$ constant. The shape of the ellipse can be altered in one of two ways. The distance between the two foci can be changed without changing the length of the string, or the length of the string can be changed without changing the positions of the foci. The distance OA (in the diagram) is known as the semi-major axis, and the distance OB as the semi-minor axis. If the two foci are made coincident (i.e. $d = 0$) the ellipse reduces to a circle.

Kepler's first law says that the planets orbit the Sun along elliptical paths, with the Sun at one focus of the elliptical orbits. The other focus has no significance in the case of planetary motion.

KEPLER'S SECOND LAW

GRAPH

AXES OF GRAPH

4.3.3 KEPLER'S SECOND LAW

Kepler's plot of the orbit of Mars, built up as it was from points separated by equal intervals of time, showed him that the planet moved with an uneven speed around its orbit (Figure 25). He was determined to find some pattern behind this 'unevenness'. He had once, much earlier in his life, published a suggestion that a planet was pushed round the Sun by spokes radiating outwards from the Sun—the force behind the spoke being smaller the longer the spoke. The idea sounds ridiculous today. Planets don't need anything to push them to keep them going. But then, 16th-century ideas of motion were rather confused.

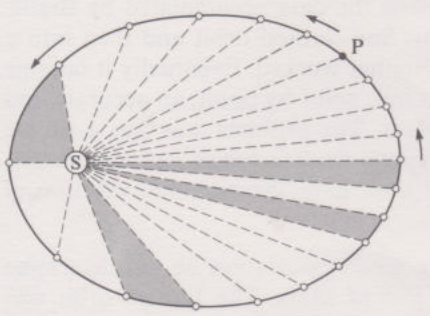


FIGURE 25 The planets orbit the Sun in elliptical paths (with the Sun at one focus of the ellipse). The positions of planet P shown here are separated by equal intervals of time, namely 1/20 of the planet's 'year'.

Nevertheless, this idea led Kepler to spot a very important relationship. He noticed that the speed of the planet's motion was uneven in such a way as to make the 'spokes' from the Sun sweep out exactly equal areas of space in equal times. All the shaded areas in Figure 25 are equal. So, if the spoke is a short one (as the planet passes near the Sun), then the speed of the planet must be large to compensate for this. Conversely, the speed is much less when the spoke is long. We now know that this holds true for all the planets: it is a general law, now called **Kepler's second law.**

> KEPLER'S SECOND LAW
>
> The 'spoke' joining the Sun to a planet sweeps out equal areas in equal times.

Notice that Kepler did not explain *why* this should be the case. He merely observed that it *was* the case. He had discovered the pattern—but he had not explained it. So, although his second law was a very useful tool for predicting the future positions of planets, it did nothing to explain the 'mysteries of space'. In fact, Kepler's second law can be explained using Newton's (more general) law of gravitation, which you will be learning about in Unit 3.

4.3.4 THE RELATIONSHIP BETWEEN ORBITAL PERIOD AND ORBITAL RADIUS

TABLE 5 Planetary data

Planet	Orbital period T (in units of Earth-years)	Orbital radius R (in units of Earth-orbital radii)
Mercury	0.24	0.39
Venus	0.62	0.72
Earth	1.00	1.00
Mars	1.88	1.52
Jupiter	11.86	5.20
Saturn	29.46	9.54

There was another 'planetary numbers' puzzle that had been worrying Kepler: what was the relationship, if any, between a planet's 'year' (i.e. the time it takes to complete one orbit round the Sun—its orbital period) and its orbital radius? He had all the data from Tycho Brahe's records, but he had not yet spotted the pattern in the data. There was one obvious regularity—the planets with the largest orbital radii* had the longest orbital times. But this is what you would expect—these planets have further to travel in one of their 'years'. Kepler was sure there must be a more definitive relationship hidden away in the data.

Table 5 shows the sort of planetary data Kepler had to work on. (Actually, these are modern data, which are slightly more accurate than those used by Kepler.) Can *you* see any relationship between T and R?

4.3.5 PLOTTING A GRAPH

If you didn't manage to spot the relationship between T and R by playing around with the numbers in Table 5, you're in good company. It took Kepler a long time to see the solution—the problem is not trivial! Nowa-

* Since the orbits are ellipses, the word 'radius' here should be interpreted to mean 'average radius'. Strictly speaking, R, as used in Kepler's laws, is the longer (semi-major) axis of the ellipse (see Figure 24).

days, however, we have a much better way of approaching it than Kepler did: we can plot a **graph**.*

We now realize that if an orbital radius of 0.39 Earth-orbital radii corresponds to an orbital time of 0.24 Earth years, and an orbital radius of 0.72 Earth-orbital radii corresponds to an orbital time of 0.62 Earth years, and so on, the easiest way to show this correspondence is on a two-dimensional diagram, in which one quantity is plotted in one direction and the second quantity is plotted in a direction at right angles to the first. Then, if there is a mathematical relationship between the two quantities, we should expect all the points to lie on a smooth curve; whereas if there is no fixed relationship, we should expect the points to be scattered with no discernible pattern.

ITQ 17 Can you think of a good argument for why a fixed relationship between two quantities should give rise to a smooth curve?

Try drawing this graph now on the grid provided in Figure 26**. The **axes** of the graph have been labelled to help you. (For the moment, just take note of the way the symbols and the units of measurement are used in the labelling; we will come back to the reason for doing it like this shortly.) You will perhaps find it easiest to plot the point corresponding to the largest values of R and T first (i.e. those for Saturn). Look along the horizontal axis until you find the value 9.54 Earth-orbital radii. Draw a faint vertical line at this value. All points on this line must have a value $R = 9.54$ Earth-orbital radii. You also know that this value of R corresponds to only one particular value of T, namely $T = 29.46$ Earth years. So look up the vertical axis until you come to this value of T. Draw a faint horizontal line at this value of T. All points on this line must have the same value of T. So, it must follow that where your faint vertical and horizontal lines cross, R equals 9.54 Earth-orbital radii, and at the same time T equals 29.46 Earth years. That is, this single point represents both the pieces of information corresponding to the planet Saturn.

Now do the same thing for all the other planets in Table 5. After you've plotted a few points, you'll probably find that you don't have to draw the intersecting lines any more—you'll be able to locate their point of intersection by eye. Incidentally, because of the scale of the axes in Figure 26, you'll find that the points corresponding to Mercury and Venus are cramped up near the origin (i.e. the point $R = 0$, $T = 0$). Yet if we expand the scale, the point corresponding to Saturn would go off the page. So, we have chosen the best compromise.

Do your points appear to lie on a smooth curve?

If not, then check to see if any of the points are plotted incorrectly. A wrongly plotted point is often clearly displaced from the curve that can be drawn through the other points. If your points lie roughly in a curve, then draw it in. Do *not* join the points up by straight lines—try to estimate the *smoothest* fit to the points, even if this means just missing one or two of them. Your curve should have no sharp kinks in it.

ITQ 18 If you, as a modern astronomer, were unexpectedly to find a planet orbiting the Sun in an orbit with an average orbital radius of 3.0 Earth-orbital radii, would you feel confident about predicting the orbital period of the planet using only the curve you have sketched in Figure 26? What do you think this orbital period would be?

* It is hard for us to realize nowadays (when graphs are such commonplace devices for displaying data) that the whole concept of graphical representation *post*dated Kepler. In fact, we owe this invention to the mathematician and philosopher, René Descartes (1596–1650).

** If you have any difficulty in plotting the graph, refer to *Into Science*, Module 7, for further assistance.

EXTRAPOLATION

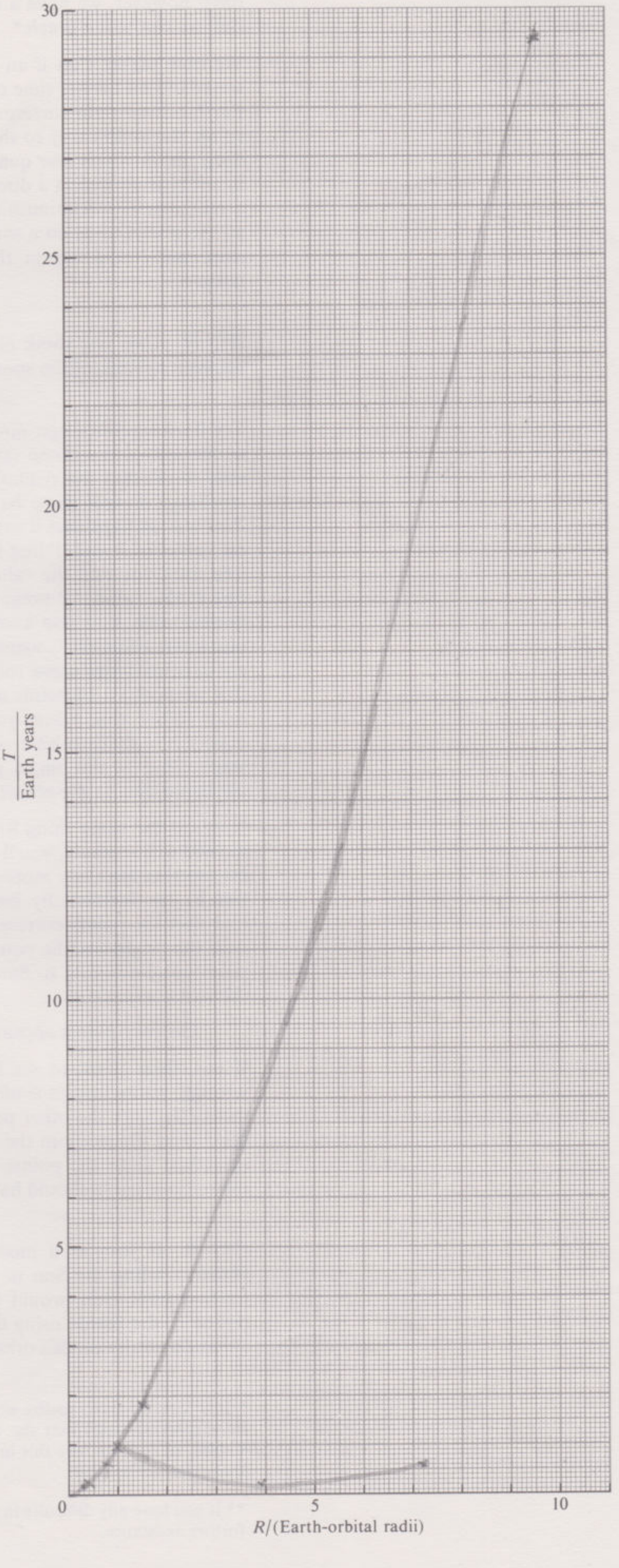


TABLE 5 Planetary data

Planet	Orbital period T (in units of Earth-years)	Orbital radius R (in units of Earth-orbital radii)
Mercury	0.24	0.39
Venus	0.62	0.72
Earth	1.00	1.00
Mars	1.88	1.52
Jupiter	11.86	5.20
Saturn	29.46	9.54

FIGURE 26 Plot points showing the correspondence between R and T for each of the planets given in Table 5 (repeated above). Note that the units of T and R are given below the division line.

ITQ 19 Did you have any trouble in getting your smooth curve to pass through the point corresponding to the Earth? What does this tell you about the Earth?

4.3.6 KEPLER'S THIRD LAW

The graph you have plotted in Figure 26 does show the relationship between T and R, but in a somewhat limited kind of way. Suppose, for instance, that you were to discover a planet with an orbital radius of 39.44 Earth-orbital radii (the value of R for the most recently discovered planet, Pluto). Could you deduce the value of T for this planet from your Figure 26 graph? The problem is that you would have to extend the graph (a process known as **extrapolation**) quite considerably beyond the range covered in Figure 26. This is tricky. Obviously, you would do your best to project the shape of the curve to higher values of T and R but, as you can probably imagine, without knowing the 'theoretical form' of the curve (i.e. without knowing the mathematical equation that relates T to R), your projection could be wildly inaccurate. This is precisely the sort of problem that faces social scientists, or governments for that matter, when they try to predict future trends in birth-rate, or unemployment, or inflation, without any knowledge of the mathematical relationships that describe these things. However, with regard to the planets, there *is* a mathematical relationship between T and R, which Kepler eventually found.

It is asking a bit much to expect you to repeat Kepler's discovery within your week's work on this Unit! However, given the relationship, you can at least verify that it really does work. What Kepler said was that if you take the period of a planet's orbit and square it and then divide the result by the cube of the radius of the planet's orbit, you will always get the same value; that is:

$$T^2/R^3 = \text{constant} \tag{20}$$

In fact, if you work in units of Earth years and Earth-orbital radii (as in Table 5), the constant you get has a numerical value of 1.00.

Table 6 repeats the data shown in Table 5, but it also has columns for T^2 and R^3 and one labelled T^2/R^3. These three additional columns have been filled in for Mercury and the Earth; the rest have been left for you to complete yourself. Before you do this, however, there are a couple of points to notice.

The first of these concerns the way the columns have been headed. The symbol for each quantity (or combination of quantities) is *divided* by the units in which that quantity (or combination) is measured. As a result, it is possible to enter the data in the body of the Table just as pure numbers: in the case of Venus, for example, the orbital period T is 0.62 Earth years and this is equivalent to saying that:

$$T/\text{Earth years} = 0.62$$

You have already met this kind of labelling system once—on the axes of the graph in Figure 26.

The second important point concerns the number of digits quoted for each quantity. Look at the data for Mercury. R has been measured in Earth-orbital radii to two digits:

$$R = 0.39 \text{ Earth-orbital radii}$$

If you use a calculator to calculate the value of R^3, you will get

$$0.39 \times 0.39 \times 0.39 = 0.059319$$

However, since the original value of R was only accurate to two digits, there is no justification for giving a value of R^3 with greater accuracy; in this context, some of the digits on the calculator display are quite simply meaningless. The value of R^3 entered in Table 6 is therefore

$$R^3 = 0.059 \text{ (Earth-orbital radii)}^3$$

KEPLER'S THIRD LAW

PROPORTIONALITY

CONSTANT OF PROPORTIONALITY

TABLE 6 Testing Kepler's third law

Planet	T Earth years	R Earth-orbital radii	T^2 (Earth years)^2	R^3 (Earth-orbital radii)^3	T^2/R^3 (Earth years)^2/ (Earth-orbital radii)^3
Mercury	0.24	0.39	0.058	0.059	0.98
Venus	0.62	0.72			
Earth	1.00	1.00	1.00	1.00	1.00
Mars	1.88	1.52			
Jupiter	11.86	5.20			
Saturn	29.46	9.54			

ITQ 20 Using your calculator, fill in the remaining gaps in Table 6. Are your results for the right-hand column consistent with Equation 20?

The relationship expressed in Equation 20 is now known as **Kepler's third law**.

KEPLER'S THIRD LAW

The square of a planet's orbital period divided by the cube of that planet's orbital radius is a constant or

$$T^2/R^3 = \text{constant} \qquad (20)^*$$

In ITQ 20 you found that the numerical value of the constant is 1.00. This, however, was only because the planetary data were expressed in units of Earth years and Earth-orbital radii. Had we used any other units, the constant would not have been unity, but Kepler's third law (Equation 20) would nevertheless have held true.

4.4 URANUS, NEPTUNE AND PLUTO

Since Kepler's day we have discovered three more planets in the Solar System—Uranus, Neptune and Pluto, and it is interesting to see whether Kepler's third law holds true for these planets as well. After all, Kepler's relationship might not be the only one that satisfies the data relating to the inner six planets: there might be an alternative formula that would also fit the series of numbers. So a seventh, eighth and ninth planet provide a way of testing the third law.

TABLE 7

Planet		R/(Earth-orbital radii)
Uranus	83.74	19.14
Neptune	165.96	30.20
Pluto	247.69	39.44

ITQ 21 Table 7 shows the orbital radii of the three planets Uranus, Neptune and Pluto. What orbital periods does Kepler's third law predict for these planets?

4.5 PROPORTIONALITY

Kepler showed that the relationship between T and R is:

$$T^2/R^3 = \text{constant} \qquad (20)^*$$

or, multiplying both sides of this equation by R^3:

$$T^2 = \text{constant} \times R^3 \qquad (21)$$

What Equation 21 says is that any value of T^2 can be found by multiplying the corresponding value of R^3 by a fixed constant; that is, all values of T^2 are in **proportion** to the corresponding values of R^3. Thus, if R^3 is doubled, T^2 is also doubled; if R^3 is multiplied by four, the corresponding value of T^2 would also be four times bigger.

In all the examples encountered so far in this Unit, the constant in Kepler's

third law has always been 1.00, in which case Equation 21 can be reduced to:

$$T^2 = R^3 \tag{22}$$

Remember, though, that this particular situation exists simply because we have chosen to express T and R in units of Earth years and Earth-orbital radii respectively (both of which are 1.00 for the Earth, of course). With different units, the constant might have been different, say 0.2 or 5 or 7.3. This would not have made any real difference, however: the T^2 values would still have been in proportion to the R^3 values. Only the so-called **constant of proportionality** would have changed.

Proportionality relationships occur so frequently in science that we use a special symbol $\propto$ as shorthand for 'is proportional to'. For Kepler's law we would say T^2 is proportional to R^3, and write this as

$$T^2 \propto R^3$$

This must be exactly equivalent to the expression

$$T^2 = kR^3 \quad \text{(where } k \text{ stands for the constant of proportionality)}$$

This is an important rule: *to convert a proportionality into an equality, a constant—called the constant of proportionality—must be included in the equation.*

ITQ 22 In Section 2 you saw that the dimensions on both sides of an equation must be the same. Kepler's third law says that:

$$T^2 = \text{constant} \times R^3$$

The dimensions of T^2 on the left-hand side of this equation are 'time squared'. The dimensions of R^3 are 'length cubed'. So how can this equation be correct?

SUMMARY OF SECTION 4

In this Section you have once again been introduced to two kinds of material: factual information relating to the motion of the planets in their orbits around the Sun, and the skills and techniques needed to analyse these planetary orbits. The information relating to the planets is encapsulated in Kepler's three laws:

1 Kepler's first law states that each planet orbits the Sun along an elliptical path, with the Sun at one of the foci of the ellipse.

2 Kepler's second law states that the imaginary line joining the Sun to an orbiting planet sweeps out equal areas in equal times.

3 Kepler's third law states that, for any orbiting planet, the ratio of T^2 to R^3 (i.e. orbital period squared to orbital radius cubed) is a constant.

In working through this Section you have also gained experience in some of the mathematical techniques that are particularly useful in analysing scientific data, notably those outlined below.

(i) It is generally possible to illustrate the relationship between two quantities by drawing a graph. One of the quantities is plotted along the vertical axis, and the other quantity along the horizontal axis. In general, a graph representing a *fixed relationship* between two quantities will take on the form of a smooth curve rather than a set of points with no obvious pattern.

(ii) An expression of proportionality can be converted into an *equality* (i.e. an equation) by introducing a constant of proportionality. For example, if $x \propto y^2$, then $x = ky^2$ where k is a constant. The *dimensions* of a constant of proportionality must be such as to retain dimensional balance within the equation. For example, if $x = ky^2$, then $k = x/y^2$, and hence the dimension of k must be equal to the dimensions of x divided by the dimensions of y^2.

5 THE MOONS OF JUPITER

5.1 THE WORK OF GALILEO

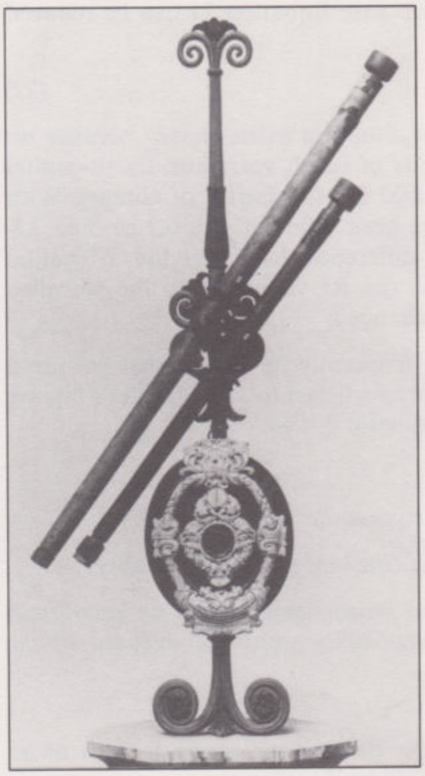


FIGURE 27 Galileo's telescope. It was probably modelled on the telescope design patented by Hans Lippershey in Holland in 1608.

The other important figure contributing to the science of astronomy at the beginning of the 17th century was the Italian, Galileo Galilei (1564–1642). Indeed, it could be said that it was Galileo who laid the foundations of modern observational astronomy by recognizing that the then newly-invented telescope—which made distant objects appear closer, and therefore larger—could be used to advantage in the study of the heavens. His first telescope magnified objects by a factor of only three, but with patience and perseverance he eventually constructed a satisfactory instrument with a magnification of $\times 30$ (Figure 27). With this new instrument he saw, for the first time, the planets not as points of light, but as luminous discs. The 'stars' were still just points in the sky (obviously much further away than the planets) but through the telescope they were brighter, further apart, and above all, far more numerous.

Perhaps Galileo's most important astronomical discovery was made with this telescope on the night of 7 January 1610. He was studying the region of the sky near the planet Jupiter when he noticed three small new 'stars'. These 'stars', together with Jupiter itself, seemed to form a straight line; one of these 'stars' was to the west of Jupiter, the other two to the east. Although he found this straight-line effect sufficiently interesting to make a sketch of the pattern, he did not think there was anything particularly strange about it. He assumed that these 'stars' were simply three more fixed and distant stars that his new telescope had brought within his view.

The surprise came the following night when he was again scanning the sky near Jupiter. The three 'stars' were still there, but now all three were to the west of Jupiter, and positioned more closely together than before. His first thought was that this shift was caused by the motion of Jupiter relative to the Earth—though he did feel that the size of the shift was surprisingly large to have taken place in only 24 hours. But then he realized that, compared with all the other stars, the shift was *in the wrong direction*—Jupiter would have had to be going the wrong way round its orbit! His curiosity was aroused. He decided to watch the 'stars' every night, and keep a record of their positions. Figure 28 shows Galileo's estimate of the positions of these 'stars' over the period 7 January to 15 January 1610. The night of 9 January must have been very frustrating for him—the sky was cloudy, and he could not see the 'stars' at all. But, on the very next night, he found the 'stars' had moved back to the east of Jupiter. This convinced him that the 'stars' themselves must be moving, thus indicating that they were not really stars after all.

What Galileo had actually found were four of the moons of Jupiter.

5.2 JUPITER'S MOONS AND KEPLER'S THIRD LAW

Kepler quickly realized that Jupiter, together with its moons, formed a kind of small-scale model of the Solar System. So, if his third law applied to the Solar System, why not to the Jupiter system also? Table 8 shows the orbital periods and orbital radii of the four innermost moons of Jupiter (in metric units this time).

TABLE 8 Orbital periods and radii for four of Jupiter's moons

moon	$\frac{T}{\text{hours}}$	$\frac{R}{\text{km}}$	$\frac{T^2}{(\text{hours})^2}$	$\frac{R^3}{(\text{km})^3}$	$\frac{T^2/R^3}{(\text{hours})^2/(\text{km})^3}$
Io	42.4	4.22×10^5	1.80×10^3	7.52×10^{16}	
Europa	85.2	6.71×10^5	7.26×10^3	3.02×10^{17}	
Ganymede	171.7	10.71×10^5	2.95×10^4	1.23×10^{18}	
Callisto	400.5	18.84×10^5	1.60×10^5	6.69×10^{18}	

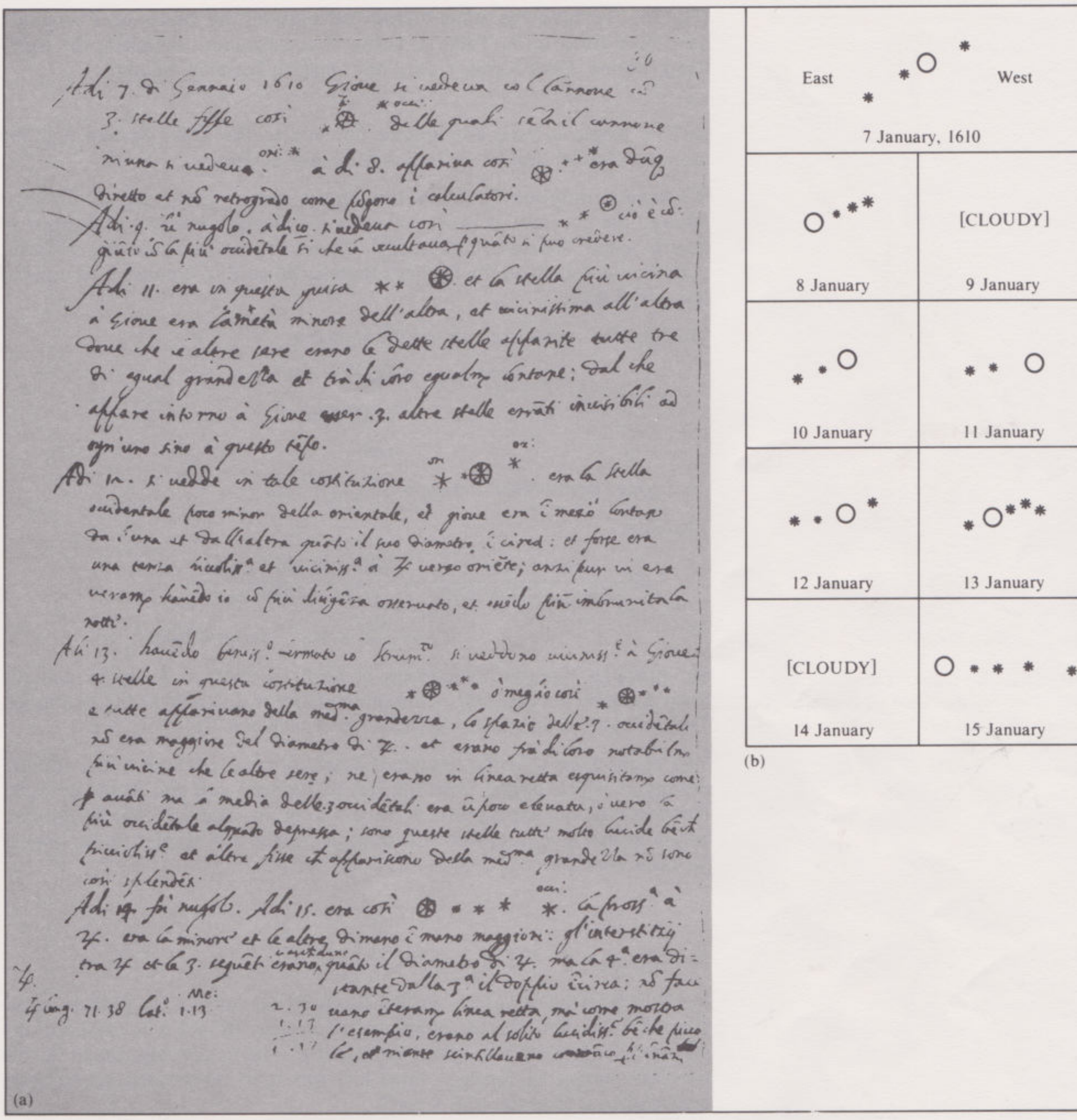


FIGURE 28 (a) A facsimile of the page from Galileo's handwritten notebook showing the positions of Jupiter's moons from 7 January to 15 January 1610.

(b) A summary of the moons' positions each night. The moons' orbits lie almost in the plane containing the line-of-sight from the Earth to Jupiter. Consequently, the moons are frequently in front of, or behind, Jupiter; that is why all four moons are not always in view simultaneously.

ITQ 23 To save you some tedious calculation, the values of T^2 and R^3 have been filled in on Table 8. Using your calculator, complete the right-hand column. Does Kepler's third law apply to Jupiter's moons?

In fact, Kepler's third law applies not just to the Solar System and the moons of Jupiter, but to *any* 'quasi-planetary' system.

We have already seen (Table 6) that, in the case of the planets of the Solar System, the constant of proportionality for Kepler's law is 1.00 in units of (Earth years)^2/(Earth-orbital radii)^3. To be able to compare this constant with the one obtained for Jupiter's moons, we need to convert it to units of (hours)^2/(km)^3.

To make the conversion, we use the facts that

$$\text{one Earth year} = (365\tfrac{1}{4} \times 24)\ \text{hours}$$
$$= 8\,766\ \text{hours}$$
$$\text{and}$$
$$\text{one Earth-orbital radius} = 1.50 \times 10^8\ \text{km}$$

Therefore

$$1.00\ \frac{(\text{Earth years})^2}{(\text{Earth-orbital radii})^3} = \frac{1.00 \times (8\,766)^2\ \text{hours}^2}{(1.50 \times 10^8)^3\ \text{km}^3}$$
$$= 2.28 \times 10^{-17}\ (\text{hours})^2/(\text{km})^3$$

So, although Equation 20 applies to both the moons of Jupiter and the Solar System, the constant of proportionality is different in the two cases. The difference is approximately a factor of 1 000 (i.e. $10^{-14}/10^{-17}$).

You might guess that the change of constant has something to do with the change of orbital-centre, from the Sun to Jupiter—and you would be right. However, the full explanation is quite a long one and will take us forward by about half a century from the days of Galileo and Kepler to the time of Sir Isaac Newton.

SUMMARY OF SECTION 5

Kepler's third law has been shown to apply not only to the planets in the Solar System, but also to the moons of Jupiter. In fact it is valid for any system of bodies orbiting a common 'centre'. The underlying reason for the universal nature of Kepler's third law was first appreciated by Newton, whose work is the subject of Unit 3.

6 TV NOTES: MEASURING—THE EARTH AND THE MOON

This programme re-examines Eratosthenes' method for determining the radius of the Earth, and the assumptions underlying the use of a lunar eclipse photograph to determine the radius of the Moon. In addition, it may give you a feel for the trials and tribulations—and the fun and enjoyment—of doing practical work in science. As you'll see, experiments don't always go as planned—even for OU academics!

The programme opens at the Bishop Walsh School in Sutton Coldfield where, with the cooperation of some of the pupils, a modern version of Eratosthenes' experiment had been set up. The school is located almost exactly 1.5° of longitude west of Greenwich; so, allowing the extra hour for British Summer Time, it was calculated that local noon would be at 1.06 p.m. However, rather than just taking one measurement of the length of the shadow at exactly local noon, the experimenter decided to follow the move-

ment of the end of the shadow over a period of about 30–40 minutes either side of local noon (Figure 29). If the length of the shadow were to remain constant over this period, then the line of marker-pins would follow the contour of the base board. However, the length of the shadow is *not* constant, because, of course, the Sun ascends higher in the sky during the morning, and is at its culmination point exactly at noon. Hence the shadow length is shortest at noon. The line of marker-pins obtained during the experiment (Figure 29) clearly demonstrates this point.

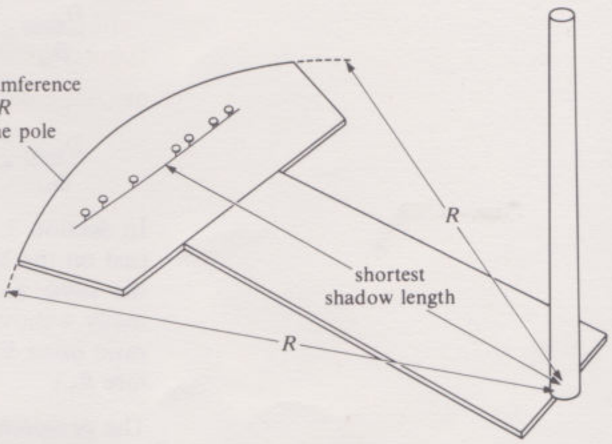


FIGURE 29 The shadow of the pole, cast on the base board, is shortest at noon.

The results of the experiment are summarized below.

At Sutton Coldfield
The length of the shadow at noon = 178 cm
The height of the pole = 318 cm

Hence the angle θ_s between the Sun's rays and the local vertical at Sutton Coldfield = 29.2°.

On the same day, the pupils of Peterhead Academy in Scotland (Peterhead is on the same line of longitude as Sutton Coldfield) found that the angle θ_p between the Sun's rays and the local vertical at Peterhead = 34.4°.

We know that the distance between Bishop Walsh School and Peterhead Academy is 552 km. The difference in angle at the two schools is $\theta_p - \theta_s = 34.4° - 29.2° = 5.2°$. Hence, an angle of 5.2° carries an arc length of 552 km.

In the Eratosthenes experiment, it is assumed that the light coming from the Sun reaches the Earth as parallel rays. However, because the Sun is not a point source of light, this assumption is not quite correct. The consequence of this for the Eratosthenes experiment is that the end of the shadow is somewhat fuzzy; this may have introduced an error of about 1% in the measurement of the length of the shadow. (*Note:* the fuzziness is nothing to do with the cloudiness of the weather.) However, when the shadow is cast over very large distances (as, for example, when the shadow of the Earth is cast on the Moon), this parallel-ray assumption can lead to quite erroneous results. Figure 30 shows how the shadow region behind the Earth is conical in shape.

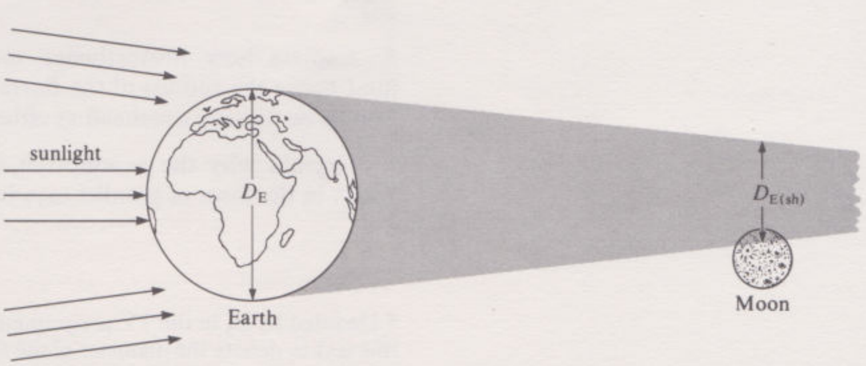


FIGURE 30 The diameter of the Earth's shadow cast on the Moon $D_{E(sh)}$ is smaller than the diameter of the Earth D_E. (Not drawn to scale.)

As explained in the programme, Aristarchus reasoned that the shadow narrows by one Moon diameter over a distance of one Moon orbital radius. That is, the diameter of the Earth's shadow $D_{E(sh)}$* cast on the Moon (i.e. one moon orbital radius away) during a lunar eclipse is less than the diameter of the Earth D_E by an amount equal to the diameter of the Moon D_M; that is:

$$D_{E(sh)} = D_E - D_M$$

Dividing by D_M:

$$\frac{D_{E(sh)}}{D_M} = \frac{D_E}{D_M} - 1$$

or

$$\frac{D_{E(sh)}}{D_M} + 1 = \frac{D_E}{D_M} \tag{23}$$

In Section 3.3 you measured the ratio of the radius of the Earth's shadow cast on the Moon to the radius of the Moon, $R_{E(sh)}/R_M$, which of course is the same as $D_{E(sh)}/D_M$. But in order to find D_M (knowing D_E) what you really want is the ratio D_E/D_M. You are now in a position to calculate this ratio using Equation 23 and hence to correct your value for D_M (and therefore R_M).

The programme closes with a brief look at the way in which modern laser lunar-ranging techniques allow scientists to make very precise measurements of the distance to the Moon.

OBJECTIVES FOR UNIT 2

Having completed your work on this Unit, you should be able to:

1 Explain the meaning of, and use correctly, all the terms flagged in the text.

2 Estimate upper and lower limits to a measurement and hence calculate a 'best value' by taking the average of the upper and lower limits. (*Experiment and ITQ 11*)

3 Calculate the upper and lower limits to a quantity that is equal to the ratio of two other quantities when the error or uncertainty in one of these quantities is much greater than that in the other. (*ITQ 12*)

4 Appreciate that the percentage error or uncertainty in the difference of two nearly equal quantities can be very large, even though the percentage error or uncertainty in these individual quantities is small.

5 Compile and understand Tables of data. (*ITQs 20 and 23*)

6 Plot a graph and interpret data presented in graphical form. (*ITQs 17–19*)

7 State Kepler's laws of planetary motion; use Kepler's third law to calculate any one of the quantities T, R or k, given the other two. (*ITQs 21 and 23*)

8 Explain how Eratosthenes' method for measuring the circumference (and hence the radius) of the Earth can be adapted so as not to require the Sun to be directly overhead at either of the two locations. (*TV*)

9 Explain why the assumption that the light from the Sun reaches the Earth in the form of parallel rays is only approximately true. (*TV*)

* Denoted by D_S in the TV programme; $D_{E(sh)}$ is used here since D_S is used earlier in this text to denote the diameter of the Sun.

10 Estimate the radius of the Earth from the radius of the Earth's shadow cast on the Moon during a lunar eclipse.

11 Rearrange mathematical equations involving addition, subtraction, multiplication, division, squares, square roots and cubes; calculate the value of a quantity from such an equation.

12 Handle reciprocals, fractions and proportions between quantities. (*ITQs 3, 20 and 23*)

13 Convert an expression of proportionality into an expression of equality by introducing a constant of proportionality.

14 Use the powers-of-ten (scientific) notation. (*ITQs 1 and 2*)

15 Use the SI units of metres, kilograms and seconds, and multiples and fractions of these units; convert physical quantities from one set of units to another, possibly using standard prefixes and/or powers of ten. (*ITQ 2*)

16 Use, appropriately, the order of magnitude symbol. (*ITQ 2*)

17 Explain what is meant by the dimensions of a physical quantity; calculate the dimensions of such a quantity given an equation relating that quantity to others of known dimensions. (*ITQs 8 and 22*)

18 Calculate the radius (or diameter) of a circle given its circumference, or vice versa. (*ITQ 4*)

19 Convert an angle from units of radians to units of degrees, and vice versa. (*ITQs 6 and 7*)

20 Use the equation arc = $R\theta$ and the small-angle approximation when appropriate; hence relate the angular size of an object to its real size and its distance from the observer. (*Experiment and ITQ 11*)

ITQ ANSWERS AND COMMENTS

ITQ 1 8.64×10^4 seconds.

There are 60 seconds in a minute, 60 minutes in one hour, and 24 hours in one solar day. Therefore, there are:

$(60 \times 60 \times 24)$ seconds in one day,

i.e. 86 400 seconds in one day

or 8.64×10^4 seconds in one day

ITQ 2 10^6 seconds.

One week contains $7 \times 8.64 \times 10^4$ seconds $= 6.048 \times 10^5$ seconds.

To within an order of magnitude, we would say that there are 10^6 seconds in one week.

You should really have been able to write down this order of magnitude answer without having to do the full calculation. After all, if you only want the answer to within an order of magnitude, why do the calculation exactly? Because 7 multiplied by 8.64 is nearer to 100 than to ten, then, in a rough calculation, 10^4 must be increased by two orders of magnitude to 10^6.

ITQ 3 24 000 miles.

360° would correspond to $360 \times (500/7.5)$ miles, that is 24 000 miles around the circumference. But remember that an angle of 360° is defined to be the angle of a complete circle. So the distance round the circle corresponding to 360° must be the complete circumference. Hence the circumference of the Earth is 24 000 miles.

ITQ 4 3 820 miles.

According to Equation 5,

$$C = 2\pi R$$

Therefore

$$R = \frac{C}{2\pi}$$

Hence, putting in values for C and π,

$$R = \frac{24\,000}{2 \times 3.14} = 3\,820 \text{ miles}$$

(rounded-off to an accuracy consistent with the data)

If you are not happy with the rearranging of Equation 5 employed in this ITQ, you should refer to Into Science, *Module 8.*

ITQ 5 6 150 km.

Eratosthenes' value for the radius of the Earth was 3 820 miles. Since 1 mile is 1.61 km, 3 820 miles is $(3\,820 \times 1.61)$ km $= 6\,150$ km.

You may be interested to know that nowadays the generally accepted value for the radius of the Earth is about 6 380 km. The difference between this value and Eratosthenes's value is $(6\,380 - 6\,150)$ km $= 230$ km. As a fraction of the accepted value, this difference is 230 km/6 380 km ≈ 0.036. Expressed as a percentage, this difference is $0.036 \times 100\% = 3.6\%$.

ITQ 6 $\pi/2$ radians.

A right angle is 90°; and 90° is one-quarter of 360°. Hence, in radians, a right angle must be equal to one quarter of 2π radians, that is:

$$90° = 2\pi/4 \text{ radians}$$
$$= \pi/2 \text{ radians}$$

ITQ 7 We know that

$$2\pi \text{ radians} = 360°$$

Therefore

$$1 \text{ radian} = \left(\frac{360}{2\pi}\right)^\circ$$
$$\approx \left(\frac{360}{2 \times 3.14}\right)^\circ \approx 57.3°$$

ITQ 8 Equation 6 says that:

$$\text{arc} = \text{radius} \times \text{angle in radians}$$

or $$\text{angle} = \text{arc/radius} \qquad (24)$$

The arc length has dimensions of length; the radius also has dimensions of length. Hence, the dimensions on the right-hand side of Equation 24 must be (length/length). That is, the right-hand side of this equation is dimensionless. If the dimensions of both sides of the equation are to balance, then the units of angle must also be dimensionless.

The radian is a dimensionless unit.

The same is true of any other unit of angular measure—the degree, for example, is also dimensionless. This is because an angle is really only a way of expressing a fraction of a rotation. So 3° really means 3/360 of a complete rotation, and 2 radians really means $2/2\pi$ of a complete rotation.

ITQ 9 3 820 miles.

The arc AS (i.e. the distance along the circumference from A to S) is given by:

$$\text{arc AS} = R_E\,\theta \text{ (from Equation 6)}$$

or equivalently

$$R_E\,\theta = \text{arc AS}$$

Dividing both sides of this equation by θ gives an expression for R_E,

i.e. $$R_E = (\text{arc AS})/\theta \qquad (25)$$

But θ must be in radians; so, since

$$360° = 2\pi \text{ radians}$$

$$1° = \frac{2\pi}{360} \text{ radians}$$

and $\theta = 7.5° = \dfrac{7.5 \times 2\pi}{360}$ radians

You don't need to work this out yet. Instead, just replace θ (in Equation 25) *by this complete fraction*—after all, they are equal. Thus

$$R_E = (\text{arc AS})/\theta \qquad (25)^*$$

$$= (500 \text{ miles}) \bigg/ \left(\frac{7.5 \times 2\pi}{360}\right) \text{radians}$$

$$= \frac{500 \times 360}{7.5 \times 2\pi} \text{ miles}$$

$$= 3\,820 \text{ miles}$$

(rounded-off to an accuracy consistent with the data)

This is exactly the same answer that you arrived at before.

If you are not happy with the way in which this fraction was rearranged, try some of the examples in Into Science, *Module 2.*

ITQ 10 13.1 cm.

Equation 9 says that for small angles:

$$\theta \text{ (in radians)} \approx \frac{\text{shadow length}}{\text{pole height}}$$

Eratosthenes' value of θ was 7.5°, or $(7.5 \times 2\pi)/360$ radians (see ITQ 9).

Substituting this value of θ into Equation 9, together with a pole height of 100 cm, gives:

$$\left(\frac{7.5 \times 2\pi}{360}\right) \text{radians} \approx \frac{\text{shadow length}}{100 \text{ cm}}$$

Multiplying both sides of this equation by 100 cm gives:

$$\left(\frac{7.5 \times 2\pi}{360}\right) \text{radians} \times 100 \text{ cm}$$

$$= \text{shadow length.}$$

that is,

$$\text{shadow length} \approx 13.1 \text{ cm}$$

ITQ 11 Probably the best thing to do in this particular case is to take the *average* of the upper and lower limit values. For example, if you estimated that the largest possible radius was 12.5 cm, and the smallest possible radius was 7.5 cm, then the average value would be

$$\frac{(12.5 + 7.5)}{2} \text{cm} = 10 \text{cm}$$

and the best estimate would be

$$(10.0 \pm 2.5) \text{ cm}$$

where adding on the 2.5 cm gives the upper limit, and subtracting the 2.5 cm gives the lower limit. (This result would be read as: R_E equals 10.0 cm, plus or minus 2.5 cm.)

Your value for R_E will probably be different to the one given here, which has simply been used to illustrate the method. *Your own result is the one you should enter in the space below the ITQ.*

ITQ 12 Suppose you found the radius of the Moon in the photograph to be (4.0 ± 0.1) cm, and the radius of the Earth (10.0 ± 2.5) cm. (*Note that these are 'made-up' figures—probably nothing like the ones you actually got.*) Then the *average* value of the ratio of the Earth radius to the Moon radius is:

$$\frac{10.0 \text{ cm}}{4.0 \text{ cm}} = 2.5$$

Now, a rough (though, in effect, somewhat pessimistic) estimate of the uncertainties associated with the ratio R_E/R_M can be found as follows. The *largest* value for the ratio would be obtained if you were to use the upper limit for the Earth radius and the lower limit for the Moon radius; that is:

$$\text{maximum possible value of } R_E/R_M = \frac{12.5}{3.9}$$

$$= 3.2$$

Conversely, the *smallest* value for the ratio is obtained by using the lower limit for R_E and the upper limit for R_M; that is:

$$\text{minimum possible value of } R_E/R_M = \frac{7.5}{4.1}$$

$$= 1.8$$

So we can write:

$$R_E = (2.5 \pm 0.7)R_M.$$

Comment: Perhaps you would like to check that you would have got more or less the same result if you had used 4.0 cm for R_M in both cases (rather than using 3.9 cm and 4.1 cm respectively). The reason why the uncertainty in R_M has very little effect on the final ratio is that the fractional (or percentage) uncertainty in R_M is swamped by the ten times larger fractional (or percentage) uncertainty in R_E.

This point is worth remembering. Whenever you combine several results, all of which have possible uncertainties associated with them, then the percentage error in the combined result will always be larger than the percentage error in any of the individual results. But if the percentage error in one measurement is much bigger than the percentage error in any of the other measurements, then the latter uncertainties can usually be neglected.

ITQ 13 In the 'made-up' example worked out in ITQs 11 and 12, the Moon was approximately two-fifths the size of the Earth; that is:

$$R_M \approx \left(\frac{2 \times 6\,200}{5}\right) \text{km} \approx 2\,480 \text{ km}$$

Remember that these values are just examples. You should, of course, calculate the radius of the Moon using

your own estimate of the ratio of the two radii. You should also estimate the upper and lower limits for this radius.

ITQ 14 1.7 cm (approx).

Remember that the equation

$$\theta_M = d/l$$

is only true if θ_M is measured in radians.
Recall that

$$1° = 2\pi/360 \text{ radians}$$

Therefore, if $l = 1$ metre, and $\theta_M = 1°$, then

$$d = l\theta_M = 1 \times \frac{2\pi}{360} \text{ metres}$$

$$\approx 1.7 \times 10^{-2} \text{ metres}$$

Hence, the diameter of the object needed to eclipse the Moon from a distance of 1 metre is *less* than 1.7 cm, i.e. less than the size of a 1p coin.

ITQ 15 Remember that

$$\text{arc} = R\theta$$

only holds true if θ is in radians.

$$3° = \frac{2\pi \times 3}{360} \text{ radians} = \frac{\pi}{60} \text{ radians}$$

Therefore, from Equation 16:

$$L_S = \frac{L_M}{(\pi/60)} \approx 20L_M$$

i.e. (according to Aristarchus) the Sun is 20 times further away from Earth than is the Moon. We now know that this ratio is about 20 times too small.

ITQ 16 If you have obtained sensible values for the distances and sizes asked for in Table 4, then you should find that:

$$L_S \approx 24\,000 \text{ Earth radii}$$

If you obtained a value for L_S between $18\,000R_E$ and $30\,000R_E$, you've done pretty well! Mind you, if you got a value *very* close to $24\,000R_E$, then you have either been very lucky, or you've been cheating!

ITQ 17 This is not an easy question to answer concisely. Perhaps the best clue is contained in the words 'a fixed relationship'. If there is a 'fixed relationship' between R and T (to take Kepler's problem as an example), then there is always only one way of calculating T if you know R (and vice versa). For example, suppose the 'fixed relationship' were $T = 2R$. Then for any value of R, no matter how big or how small R might be, T would always be found by multiplying R by two.

Now consider one specific value of R and its corresponding value of T. Suppose R is increased by a small amount. Obviously T will increase by a correspondingly small amount (related to R through the fixed relationship $T = 2R$).

Successive increases in R will produce corresponding increases in T, that is, as R changes smoothly, T also changes smoothly. So whenever there is a *fixed relationship* between two quantities, the curve showing this relationship will, in general, be a smooth curve.

No such restriction will apply if there is *no* fixed relationship between the two quantities. In this situation, when R is changed by a small amount, it is quite possible for T to take on any value at all. Consequently the points could be scattered in no discernible pattern across the graph paper.

ITQ 18 If you obtained a smooth curve for Figure 26 (and you should have done), you should be feeling fairly confident that there is some form of fixed relationship between R and T. If this is the case, then you can deduce pairs of values for R and T other than those corresponding to the six planets you have plotted.

In other words, you would be able to say that, if a planet had an orbital radius R of 3.0 Earth-orbital radii, then it must also have a period T of about 5.2 Earth years. How do you know that it is 5.2? Simply by reading from your graph that T value corresponding to an R value of 3.0 Earth-orbital radii. That is, read vertically up the graph at $R = 3.0$ Earth-orbital radii until you get to the curve, and then note the T value corresponding to this point on the curve. This is shown in Figure 31. This process of finding pairs of values on graph (here, $T = 5.2$ and $R = 3.0$) lying *between* two plotted values, is known as *interpolation*.

ITQ 19 The curve you have drawn should pass easily through the point corresponding to the Earth, which fits nicely between the points corresponding to Venus and Mars. In Kepler's terms, this tells you that the Earth is the same kind of 'object' as the other planets. Copernicus was right—there is nothing special about the Earth. The relationship between T and R, which you have found in graphic form in Figure 26, is a relationship that applies to all planets orbiting the Sun including the Earth.

TABLE 9 For ITQ 20.

	T (Earth years)	R (Earth-orbital radii)	T^2 (Earth years)^2	R^3 (Earth-orbital radii)^3	T^2/R^3 (Earth years)^2/(Earth-orbital radii)^3
Mercury	0.24	0.39	0.058	0.059	0.98
Venus	0.62	0.72	0.384	0.373	1.03
Earth	1.00	1.00	1.00	1.00	1.00
Mars	1.88	1.52	3.53	3.51	1.01
Jupiter	11.86	5.20	141	141	1.00
Saturn	29.46	9.54	868	868	1.00

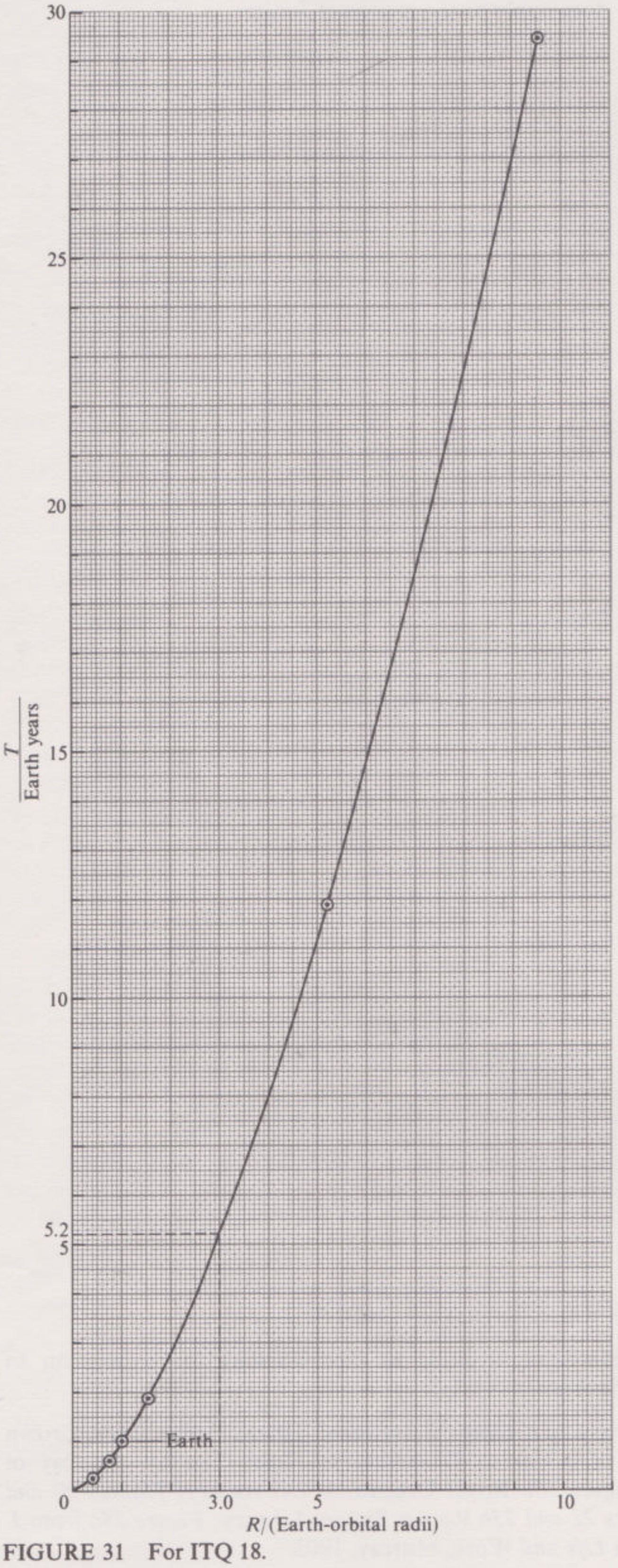


FIGURE 31 For ITQ 18.

ITQ 20 As you can see from the final column of Table 9, T^2 divided by R^3 comes to almost exactly unity for all the planets. The slight deviation is never more than a few per cent (a few parts in one hundred) and can be attributed to experimental uncertainty.

ITQ 21 83.74, 166.0 and 247.7 Earth years.

Kepler's third law says that:

$$T^2/R^3 = \text{constant} \tag{20}*$$

If T is measured in Earth years, and R in Earth-orbital radii, then the constant is exactly 1.00. Hence

$$T^2 = R^3 \tag{22}*$$

so

$$T = \sqrt{R^3} \tag{26}$$

Alternatively, using the fractional index notation, we can write

$$T = (R^3)^{1/2} \tag{27}$$

(A number to the power one-half means the square root of the number; a number to the power one-third means the cube root of the number; a number to the power $1/n$ means the nth root of the number.)

If you are not happy with this notation you should look at Into Science, *Modules 3 and 4.*

Using the radius values of Table 7 in Equation 27 gives:

$$\text{Uranus: } T = (19.14^3)^{1/2} \approx (7\,011.7)^{1/2} \approx 83.74 \text{ Earth years}$$

(rounded-off to four digits)

$$\text{Neptune: } T = (30.20^3)^{1/2} \approx (27\,544)^{1/2} \approx 166.0 \text{ Earth years}$$

(rounded-off to four digits)

$$\text{Pluto: } T = (39.44^3)^{1/2} \approx (61\,349)^{1/2} \approx 247.7 \text{ Earth years}$$

These are the values of T predicted by Kepler's third law. The modern measurements for these values of T are 83.74, 165.95 and 247.69 Earth years, respectively. It looks as though Kepler was right! You should now feel confident enough to put aside your graph paper and use the formula instead.

ITQ 22 Trust the dimensional argument! The only way to make the dimensions on both sides of the equation balance, is to acknowledge that the constant in the equation must also have dimensions. Thus, writing the equation in terms of its dimensions:

$$(\text{time})^2 = (\text{dimensions of the constant}) \times (\text{length})^3$$

This equation balances if the dimensions of the constant are those of (time)^2/(length)^3, since the dimensions of the right-hand side can then be written as:

$$\frac{(\text{time})^2}{(\text{length})^3} \times (\text{length})^3 = (\text{time})^2$$

which is the same as on the left-hand side.

The *units* of the constant (as used in ITQ 20—see Table 9 in ITQ 20 answer) are (Earth years)^2/(Earth-orbital radii)^3. In SI units the constant would be in s^2/m^3.

ITQ 23 The value of the ratio T^2/R^3 is roughly the same for all four moons of Jupiter, about

$$2.4 \times 10^{-14}\ (\text{hours})^2/(\text{km})^3$$

Equation 20 is therefore satisfied: Kepler's third law does apply to the moons of Jupiter.

ACKNOWLEDGEMENTS

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